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Rigorous link between the conductivity and elastic moduli of fibre-reinforced composite materials

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We derive rigorous cross-property relations linking the effective transverse electrical conductivity σ_* and the effective transverse elastic moduli of any transversely isotropic, two-phase 'fibre-reinforced' composite whose phase boundaries are cylindrical surfaces with generators parallel to one axis. Specifically, upper and lower bounds are derived on the effective transverse bulk modulus κ_* in terms of σ_* and on the effective transverse shear modulus μ_* in terms of σ_* . These bounds enclose certain regions in the $\sigma_*-\kappa_*$ and $\sigma_*-\mu_*$ planes, portions of which are attainable by certain microgeometries and thus optimal. Our bounds connecting the effective conductivity σ_* to the effective bulk modulus κ_* apply as well to anisotropic composites with square symmetry. The implications and utility of the bounds are explored for some general situations, as well as for specific microgeometries, including regular and random arrays of circular cylinders, hierarchical geometries corresponding to effective-medium theories, and checkerboard models. It is shown that knowledge of the effective conductivity can yield sharp estimates of the effective elastic moduli (and vice versa), even for infinite phase contrast.

1. Introduction and cross-property statements

An intriguing fundamental as well as practical question in the study of heterogeneous materials is the following: Can different properties of the medium be *rigorously* linked to one another? Such cross-property relations become especially useful if one property is more easily measured than another property. Since the effective properties of random media reflect certain morphological information about the medium, one might expect that one could extract useful information about one effective property given an accurate (experimental or theoretical) determination of another property, even when their respective governing equations are uncoupled. Although recent progress has been made in the establishment of rigorous cross-property relations for a variety of properties (Prager 1969; Milton 1984; Berryman & Milton 1988; Torquato 1990, 1992; Avellaneda & Torquato 1991; Cherkaev & Gibiansky 1992; Gibiansky & Torquato 1993), this subject remains fertile ground for research.

In this article we focus our attention on deriving cross-property relations between the effective transverse conductivity and effective transverse elastic moduli of transversely isotropic, two-phase 'fibre-reinforced' composites whose phase boundaries are cylindrical surfaces with generators parallel to one axis. Thus, the problem reduces to

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studying two-dimensional, two-phase, isotropic composites (i.e. plane elasticity and conductivity) with effective properties that are identical to the corresponding transverse properties of fibre-reinforced composites (Christensen 1979). For an isotropic medium, the material is characterized by three scalar constants: conductivity σ , bulk modulus κ , and shear modulus μ . The properties of the first and second phases are described by the triples $(\sigma_1, \kappa_1, \mu_1)$, $(\sigma_2, \kappa_2, \mu_2)$, respectively. Similarly, the triple $(\sigma_*, \kappa_*, \mu_*)$ denotes the corresponding effective properties which depend on the phase properties, phase volume fractions f_1 and $f_2 = 1 - f_1$, and on the microstructure. For every fixed microstructure, one may calculate the effective properties. Therefore, any composite corresponds to some point $(\sigma_*, \kappa_*, \mu_*)$ in a three-dimensional σ_* - κ_* - μ_* space. As the microstructure is varied, this point moves and traces out some region in this space. It is of great interest to know the whole set of such points that corresponds to composites of all possible structures. However, this is a very difficult problem in general and we do not attempt to resolve it here. Instead our objective is to find some bounds on this set. Moreover, we restrict our attention to the bounds of the projection of this set onto the σ_* - κ_* and σ_* - μ_* planes.

Before discussing the main results of this paper, it is useful first to describe some related results. Using variational principles, Milton (1985) showed that, for arbitrary d -dimensional, isotropic, two-phase media, if the phase bulk moduli κ_i equal the phase conductivities σ_i and phase Poisson's ratios ν_i are positive, then the effective bulk modulus κ_* is bounded from above by the effective conductivity σ_* . It is simple to extend Milton's result to the more general situation in which $\kappa_2/\kappa_1 \leq \sigma_2/\sigma_1$ (Torquato 1992). Specifically, for d -dimensional isotropic two-phase media of arbitrary topology having positive phase Poisson's ratios $\nu_i = (\kappa_i - \mu_i)/(\kappa_i + \mu_i)$, the following dimensionless relation holds:

$$\kappa_*/\kappa_1 \leq \sigma_*/\sigma_1, \quad (1.1)$$

where $\kappa_2/\kappa_1 \leq \sigma_2/\sigma_1$. Using this result, Torquato (1992) derived an upper bound on the effective shear modulus μ_* in terms of σ_* and the effective Poisson's ratio $\nu_* = (\kappa_* - \mu_*)/(\kappa_* + \mu_*)$. In the case of $d = 2$, this expression reads

$$\frac{\mu_*}{\kappa_1} \leq \frac{\sigma_* (1 - \nu_*)}{\sigma_1 (1 + \nu_*)}, \quad (1.2)$$

where again $\kappa_2/\kappa_1 \leq \sigma_2/\sigma_1$. It is important to note that relations (1.1) and (1.2) are valid for arbitrary volume fraction $f_1 = 1 - f_2$. We shall come back to these expressions later in the paper. Berryman & Milton (1988) found cross-property relations for σ_* - κ_* and σ_* - μ_* relations for *three-dimensional isotropic composites* in terms of the conductivity by eliminating geometrical parameters involved in three-point bounds on the properties. One can get corresponding results for the problem under study. We mention, however, that the application of the Berryman-Milton procedure in the two-dimensional space yields cross-property bounds which generally are not as sharp as the ones derived in the present work. This suggests that their three-dimensional results also can be improved. This will be the subject of a future paper.

Our major findings are that we have obtain the sharpest known bounds on the sets of pairs σ_* - κ_* and σ_* - μ_* that correspond to two-dimensional, two-phase, isotropic composites of all possible microgeometries at a prescribed or arbitrary volume fraction f_1 by using the so-called *translation method*. These bounds enclose certain regions in the σ_* - κ_* and σ_* - μ_* planes. Particular boundaries of these regions are realizable by certain microgeometries and thus are optimal bounds in these instances. It is

important to emphasize that our results for the effective conductivity and bulk modulus are not restricted to isotropic composites only but apply as well to *anisotropic composites with square symmetry* (e.g. checkerboard models).

We note that the determination of the electrical conductivity σ_* is mathematically equivalent to finding either the *thermal conductivity, dielectric constant, magnetic permeability or diffusion coefficient*. Thus our cross-property relations connect the elastic moduli to any of these properties as well.

Our cross-property bounds were already stated in a short letter (Gibiansky & Torquato 1993). However, the proof of these bounds were not given and few applications were considered. The major aims of the present work are to provide detailed proofs of the cross-property relations and to apply them for some general situations as well as for specific microgeometries, including regular and random arrays of circular cylinders, hierarchical geometries corresponding to effective-medium theories, and checkerboard-type models.

Let us now state our main results, namely, cross-property bounds on the sets of pairs (σ_*, κ_*) and (σ_*, μ_*) that correspond to two-dimensional, two-phase, isotropic composites of all possible microgeometries at a prescribed volume fraction. To describe the bounds, we first need to introduce some notation. Let $F(d_1, d_2, f_1, f_2, y)$ be the following function:

$$F(d_1, d_2, f_1, f_2, y) = f_1 d_1 + f_2 d_2 - \frac{f_1 f_2 (d_1 - d_2)^2}{f_2 d_1 + f_1 d_2 + y}. \quad (1.3)$$

For simplicity of notation we will further omit the first four arguments and will use the brief notation $F(d_1, d_2, f_1, f_2, y) = F_d(y)$.

Remark 1.1. This function is a scalar variant of the inverse Y -transformation. The definition and properties of the Y -transformation will be discussed in §2.

Now let σ_{1*}, σ_{2*} denote the expressions,

$$\sigma_{1*} = F_\sigma(\sigma_1), \quad \sigma_{2*} = F_\sigma(\sigma_2). \quad (1.4)$$

Similarly, let κ_{1*}, κ_{2*} denote the expressions,

$$\kappa_{1*} = F_\kappa(\mu_1), \quad \kappa_{2*} = F_\kappa(\mu_2) \quad (1.5)$$

and $\mu_{1*}, \mu_{2*}, \mu_{3*}, \mu_{4*}$, denote the expressions,

$$\mu_{1*} = F_\mu(\kappa_1 \mu_1 / (\kappa_1 + 2\mu_1)), \quad \mu_{2*} = F_\mu(\kappa_2 \mu_2 / (\kappa_2 + 2\mu_2)), \quad (1.6)$$

$$\mu_{3*} = F_\mu(\kappa_2 \mu_1 / (\kappa_2 + 2\mu_1)), \quad \mu_{4*} = F_\mu(\kappa_1 \mu_2 / (\kappa_1 + 2\mu_2)). \quad (1.7)$$

Moreover, let the harmonic average of the phase bulk moduli be defined by

$$\kappa_h = [f_1/\kappa_1 + f_2/\kappa_2]^{-1} = F_\kappa(0). \quad (1.8)$$

Remark 1.2. The formulas (1.4) and (1.5) coincide with the two-dimensional variant of the upper and lower Hashin–Shtrikman bounds on the effective conductivity (see Hashin & Shtrikman 1962) and effective bulk modulus (see Hashin & Shtrikman 1963; Hashin 1965) of isotropic composites, respectively. Formulas (1.6) and (1.7) coincide with the two-dimensional Hashin–Shtrikman–Walpole bounds on the shear modulus (see Hashin & Shtrikman 1963; Hashin 1965; Walpole 1966, 1969). These bounds were also obtained by Hill (1964).

The bounds that we found are given by segments of hyperbolas in the $\sigma_*-\kappa_*$ and $\sigma_*-\mu_*$ planes with asymptotes that are parallel to the axes $\sigma_* = 0$, $\kappa_* = 0$ or $\sigma_* = 0$, $\mu_* = 0$. For this reason we mention that every such hyperbola in the x_*-y_* plane can be described by the equation

$$A(x_* - x_0)(y_* - y_0) = 1. \quad (1.9)$$

It can be defined by three points that it passes through. We denote by

$$\text{Hyp}[(x_1, y_1), (x_2, y_2), (x_3, y_3)]$$

the segment of such hyperbola that passes through the point (x_1, y_1) , (x_2, y_2) , and when extended, passes through the point (x_3, y_3) . It may be parametrically described in the x_*-y_* plane as follows:

$$\left. \begin{aligned} x_* &= \gamma x_1 + (1 - \gamma)x_2 - \frac{\gamma(1 - \gamma)(x_1 - x_2)^2}{(1 - \gamma)x_1 + \gamma x_2 - x_3}, \\ y_* &= \gamma y_1 + (1 - \gamma)y_2 - \frac{\gamma(1 - \gamma)(y_1 - y_2)^2}{(1 - \gamma)y_1 + \gamma y_2 - y_3}, \end{aligned} \right\} \quad (1.10)$$

where $\gamma \in [0, 1]$. Now we are ready to state our main results.

(a) *Conductivity-bulk modulus bounds*

Statement 1.1. To find cross-property bounds on the set of the pairs (σ_*, κ_*) for any isotropic composite at a fixed volume fraction $f_1 = 1 - f_2$, one should inscribe in the conductivity-bulk modulus plane the segments of the following four hyperbolas:

$$\text{Hyp}[(\sigma_{1*}, \kappa_{1*}), (\sigma_{2*}, \kappa_{2*}), (\sigma_1, \kappa_h)], \quad \text{Hyp}[(\sigma_{1*}, \kappa_{1*}), (\sigma_{2*}, \kappa_{2*}), (\sigma_2, \kappa_h)],$$

$$\text{Hyp}[(\sigma_{1*}, \kappa_{1*}), (\sigma_{2*}, \kappa_{2*}), (\sigma_1, \kappa_1)], \quad \text{Hyp}[(\sigma_{1*}, \kappa_{1*}), (\sigma_{2*}, \kappa_{2*}), (\sigma_2, \kappa_2)].$$

The outermost pair of these curves gives us the desired bounds (see figure 1).

Remark 1.3. Statement 1.1 connecting the effective conductivity to the effective bulk modulus is not restricted to isotropic composites only but applies as well to anisotropic composites with square symmetry.

Figure 1 depicts conductivity-bulk modulus bounds for the following values of the parameters:

$$\sigma_2/\sigma_1 = 20, \quad \kappa_2/\kappa_1 = 20, \quad \nu_1 = \nu_2 = 0.3, \quad f_1 = 0.2. \quad (1.11)$$

The corner points $A = (\sigma_{1*}, \kappa_{1*})$, and $B = (\sigma_{2*}, \kappa_{2*})$ of the set enclosed by the bounds are *optimal* because they correspond to assemblages of coated circles (Hashin & Shtrikman 1963) as well as to isotropic matrix laminate composites (Francfort & Murat 1986). The hyperbolas $\text{Hyp}[(\sigma_{1*}, \kappa_{1*}), (\sigma_{2*}, \kappa_{2*}), (\sigma_1, \kappa_1)]$ (curve 1 of figure 1), and $\text{Hyp}[(\sigma_{1*}, \kappa_{1*}), (\sigma_{2*}, \kappa_{2*}), (\sigma_2, \kappa_2)]$ (curve 2 of figure 1) correspond to the assemblages of doubly coated spheres or to doubly coated matrix laminate composites (see Cherkaev & Gibiansky 1992; Gibiansky & Milton 1993). Depending upon the values of the parameters, one of these curves always forms part of the bound (upper bound of figure 1). Thus, this is an optimal bound because there exist composites that realize it. At the moment we do not know any structures that realize the other two segments of hyperbolas (curves 3 and 4 of figure 1).

Note that unlike the bound (1.1), our results include information about the phase

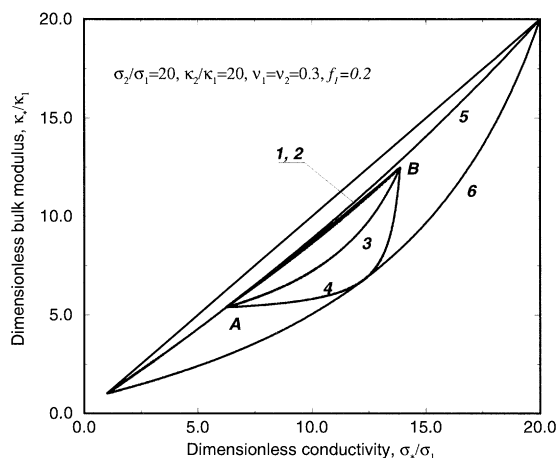


Figure 1. Cross-property bounds in the conductivity-bulk modulus plane. The internal region (bounded by curves 1 and 4) represents the bounds for fixed volume fraction. Curves 1, 2, 3 and 4 are the segments of the hyperbolas described in statement 1.1. On the scale of the figure, curves 1 and 2 appear to coincide. The larger region (bounded by curves 5 and 6) represents the bounds for arbitrary volume fraction. The unmarked straight line is the bound (1.1).

volume fractions. In order to obtain bounds for arbitrary volume fraction, one can take the union of the sets defined by our bounds over the phase volume fractions. This union is bounded by the curve 5 and 6 of figure 1. The unmarked straight line of figure 1 corresponds to the upper bound of relation (1.1). This bound is optimal and coincides with our new bound when $\sigma_2/\sigma_1 = \kappa_2/\kappa_1$ and the Poisson's ratios of the phases are equal to zero (i.e. $\mu_1/\kappa_1 = \mu_2/\kappa_2 = 1$). In general our volume-fraction independent upper bound is more restrictive than (1.1).

(b) Conductivity-shear modulus bounds

Statement 1.2. To find cross-property bounds on the set of the pairs (σ_*, μ_*) for any composite at fixed volume fraction $f_1 = 1 - f_2$, one should inscribe in the conductivity-shear modulus plane the segments of the following four hyperbolas:

$$\text{Hyp}[(\sigma_{1*}, \mu_{1*}), (\sigma_{2*}, \mu_{3*}), (\sigma_1, \mu_1)], \quad \text{Hyp}[(\sigma_{1*}, \mu_{1*}), (\sigma_{2*}, \mu_{3*}), (\sigma_2, \mu_2)],$$

$$\text{Hyp}[(\sigma_{1*}, \mu_{4*}), (\sigma_{2*}, \mu_{2*}), (\sigma_1, \mu_1)], \quad \text{Hyp}[(\sigma_{1*}, \mu_{4*}), (\sigma_{2*}, \mu_{2*}), (\sigma_2, \mu_2)],$$

and the segments of two straight lines:

$$\sigma_* = \sigma_{1*}, \quad \mu_* \in [\mu_{1*}, \mu_{4*}], \quad \text{and} \quad \sigma_* = \sigma_{2*}, \quad \mu_* \in [\mu_{2*}, \mu_{3*}].$$

The outermost of these curves give us the desired bounds (see figure 2).

Figure 2 illustrates the conductivity-shear modulus bounds for the phase moduli and volume fractions as specified by (1.11). We see that the cross-property bounds in the $\sigma_*-\mu_*$ plane are represented by a curvilinear trapezium. The straight sides AB and CD are given by the Hashin-Shtrikman bounds on the effective conductivity. The other two curvilinear sides (new bounds that are denoted as curves 1 and 4 of figure 2) are the hyperbola segments. Observe that the two corner points $A = (\sigma_{1*}, \mu_{1*})$, and $C = (\sigma_{2*}, \mu_{2*})$ of the set enclosed by the bounds correspond to the matrix laminate

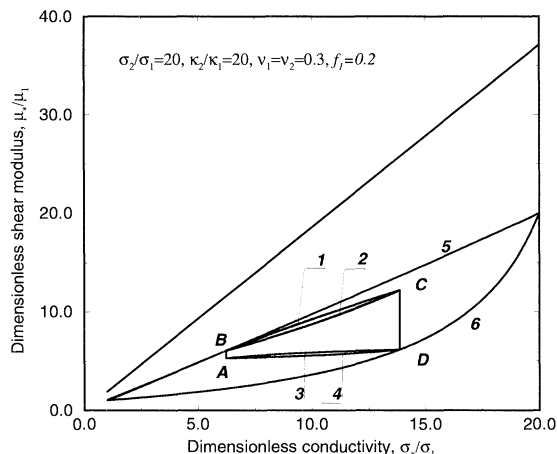


Figure 2. Cross-property bounds in the conductivity-shear modulus plane. The internal trapezium ABCD (B and C joined by curve 1 and A and D joined by curve 4) represents the bounds for fixed volume fraction. Curves 1, 2, 3 and 4 are the segments of the hyperbolas described in statement 1.2. The larger internal lens-shaped region (bounded by curves 5 and 6) represents the bounds for arbitrary volume fraction. The unmarked straight line is the bound (1.2) with $\nu_* = 0$.

composites that realize the Hashin–Shtrikman bounds for elasticity and conductivity and thus are optimal (Francfort & Murat 1986). The other two points $B = (\sigma_{1*}, \mu_{4*})$ and $D = (\sigma_{2*}, \mu_{3*})$ correspond to the structures that could realize the Walpole bounds on the shear modulus of a composite. It is unknown at the moment whether there exist such geometries. Note that there exist composites having the property pair (σ_*, μ_*) that lie on the vertical sides of the trapezium (see figure 2). They are polycrystals made of square symmetric matrix laminate composites (Cherkaev & Gibiansky 1984).

One can again take the union of the sets defined by our bounds over the phase volume fractions, to obtain cross-property bounds for arbitrary volume fraction (set bounded by the curves 5 and 6 of figure 2). The unmarked straight line of figure 2 corresponds to the upper bound (1.2) with $\nu_* = 0$ (i.e. a weaker form of equation (1.2)), which does not incorporate volume fraction information. Our volume-fraction-independent upper bound is more restrictive than the weak form of (1.2) (with $\nu_* = 0$).

In summary, the effective elastic moduli and conductivities of composites are not independent but are, in fact, connected through the microstructure. However, this relation is not one-to-one because composites of different microgeometries may possess the same conductivities but different elastic moduli, and vice versa. The cross-property bounds given here show how the values of the effective bulk or shear modulus (conductivity) of the composite are constrained assuming that the phase properties and volume fractions are fixed and the effective conductivity (bulk or shear modulus) is known.

The detailed proofs of the above cross-property bounds and the methodology leading to them are described in the ensuing sections. Application of these bounds for some general situations as well as for specific microstructures are given in the last

section of the paper. The reader who is interested in the results only can skip §§ 2–5 and go directly to § 6.

2. Local and homogenized equations

In this section we describe the local and homogenized equations of plane elasticity and conductivity, and introduce relevant notations.

(a) Basic equations and notations

We deal with the plane problem of the elasticity and the plane conductivity problem. The elastic state of the body is described by the local relations,

$$\nabla \cdot \boldsymbol{\tau} = 0, \quad \boldsymbol{\epsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad \boldsymbol{\tau} = (\boldsymbol{\tau})^T, \quad \boldsymbol{\tau} = \mathbf{C} : \boldsymbol{\epsilon}, \quad (2.1)$$

where \mathbf{u} is the displacement vector, $\boldsymbol{\epsilon}$ and $\boldsymbol{\tau}$ are the strain and stress tensors, respectively, and \mathbf{C} is the stiffness tensor.

It is convenient to introduce the following orthonormal basis in the space of the second-order tensors (for the elasticity problem we use the representation by Lurie & Cherkav (1984*b*, 1986*b*); see also Cherkav & Gibiansky (1993) and references therein):

$$\mathbf{a}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{a}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{a}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{a}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.2)$$

The tensors \mathbf{a}_1 , \mathbf{a}_2 , \mathbf{a}_3 form the orthonormal basis of the space of the symmetric second-order tensors. Strain and stress tensors have the following representation in this basis,

$$\boldsymbol{\epsilon} = \sum_{i=1}^3 \epsilon_i \mathbf{a}_i, \quad \boldsymbol{\tau} = \sum_{i=1}^3 \tau_i \mathbf{a}_i, \quad (2.3)$$

where the coefficients ϵ_i , τ_i are given by

$$\left. \begin{aligned} \epsilon_1 = \boldsymbol{\epsilon} : \mathbf{a}_1 &= \frac{1}{\sqrt{2}}(\epsilon_{11} + \epsilon_{22}), & \epsilon_2 = \boldsymbol{\epsilon} : \mathbf{a}_2 &= \frac{1}{\sqrt{2}}(\epsilon_{11} - \epsilon_{22}), & \epsilon_3 = \boldsymbol{\epsilon} : \mathbf{a}_3 &= \sqrt{2} \epsilon_{12}, \\ \tau_1 = \boldsymbol{\tau} : \mathbf{a}_1 &= \frac{1}{\sqrt{2}}(\tau_{11} + \tau_{22}), & \tau_2 = \boldsymbol{\tau} : \mathbf{a}_2 &= \frac{1}{\sqrt{2}}(\tau_{11} - \tau_{22}), & \tau_3 = \boldsymbol{\tau} : \mathbf{a}_3 &= \sqrt{2} \tau_{12}, \end{aligned} \right\} \quad (2.4)$$

and ϵ_{ij} , τ_{ij} are the elements of the matrices $\boldsymbol{\epsilon}$ and $\boldsymbol{\tau}$ in the Cartesian basis \mathbf{i} , \mathbf{j} .

Remark 2.1. The symbol $:$ denotes contraction with regards to two indices, i.e.

$$\mathbf{a} : \mathbf{b} = \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} b_{ji}; \quad \mathbf{a} = \mathbf{D} : \mathbf{b} \quad \text{if} \quad a_{ij} = \sum_{k=1}^2 \sum_{l=1}^2 D_{ijkl} b_{lk}, \quad i, j = 1, 2. \quad (2.5)$$

In the basis (2.2), Hooke's law for an isotropic material has the form (Atkin & Fox 1990)

$$\tau_1 = 2\kappa\epsilon_1, \quad \tau_2 = 2\mu\epsilon_2, \quad \tau_3 = 2\mu\epsilon_3. \quad (2.6)$$

Here $\kappa = \kappa(\mathbf{x})$ and $\mu = \mu(\mathbf{x})$ are the bulk and shear moduli, respectively, of the plane elasticity (plane-strain) problem at the point $\mathbf{x} = (x_1, x_2)$.

Remark 2.2. The plane-strain bulk modulus κ is expressed as

$$\kappa = k + \frac{1}{3}\mu, \quad (2.7)$$

in terms of the bulk modulus k of three-dimensional theory (Christensen 1979). The plane shear modulus μ does not differ from the transverse shear modulus used in the three-dimensional theory.

Remark 2.3. If the composite is a thin plate, then we consider the plane stress problem. In this case the plane stress bulk modulus κ is expressed as

$$\kappa = \frac{9k\mu}{3k + 4\mu} \quad (2.8)$$

(Christensen 1979; Eischen & Torquato 1993). The plane stress shear modulus μ again does not differ from the shear modulus used in the three-dimensional theory.

The isotropic stiffness tensor $\mathbf{C}(\kappa, \mu)$ is represented by the diagonal matrix

$$\mathbf{C}(\kappa, \mu) = \begin{pmatrix} 2\kappa & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix}. \quad (2.9)$$

The elastic energy density can be written either as a quadratic form of strains

$$W_\epsilon(\boldsymbol{\epsilon}) = \boldsymbol{\epsilon} : \mathbf{C} : \boldsymbol{\epsilon}, \quad (2.10)$$

or as a quadratic form of stresses

$$W_\tau(\boldsymbol{\tau}) = \boldsymbol{\tau} : \mathbf{S} : \boldsymbol{\tau}, \quad (2.11)$$

where

$$\mathbf{S} = \mathbf{C}^{-1} = \begin{pmatrix} 1/2\kappa & 0 & 0 \\ 0 & 1/2\mu & 0 \\ 0 & 0 & 1/2\mu \end{pmatrix} \quad (2.12)$$

is the compliance tensor.

Following Cherkaev & Gibiansky (1993), we also introduce the non-symmetric matrix $\boldsymbol{\zeta} = \nabla \mathbf{u}$ of the gradient of displacement vector \mathbf{u}

$$\boldsymbol{\zeta} = \nabla \mathbf{u}, \quad \boldsymbol{\zeta} = \sum_{i=1}^4 \zeta_i \mathbf{a}_i, \quad \zeta_i = \boldsymbol{\zeta} : \mathbf{a}_i, \quad i = 1, 2, 3, 4. \quad (2.13)$$

Note that the first three components of tensors $\boldsymbol{\zeta}$ and $\boldsymbol{\epsilon}$ coincide, i.e.

$$\zeta_i = \epsilon_i, \quad i = 1, 2, 3, \quad (2.14)$$

and the fourth component of $\boldsymbol{\zeta}$ ($\zeta_4 \neq \epsilon_4 = 0$) does not effect the equations of elasticity. Specifically, the elastic energy density (2.10) as a function of the tensor $\boldsymbol{\zeta}$ becomes

$$W_\zeta(\boldsymbol{\zeta}) = \boldsymbol{\zeta} : \mathbf{C}' : \boldsymbol{\zeta}, \quad (2.15)$$

where the (4×4) matrix \mathbf{C}' has the diagonal form

$$\mathbf{C}' = \begin{pmatrix} 2\kappa & 0 & 0 & 0 \\ 0 & 2\mu & 0 & 0 \\ 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.16)$$

in the introduced basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$.

The conductivity problem is described by the local relations,

$$\nabla \cdot \mathbf{j} = 0, \quad \mathbf{j} = \boldsymbol{\Sigma} \cdot \mathbf{e}, \quad \mathbf{e} = -\nabla \phi, \quad (2.17)$$

where ϕ is the electrical potential, \mathbf{j} and \mathbf{e} are the current and the electrical fields, respectively. The tensor $\boldsymbol{\Sigma}$ of the electrical conductivity of an isotropic material has the form,

$$\boldsymbol{\Sigma} = \sigma I, \quad (2.18)$$

where σ is a conductivity constant of an isotropic media and I is the (2×2) unit matrix.

The electrostatic energy density can be presented as a quadratic form in either the electric field,

$$W_e(\mathbf{e}) = \mathbf{e} \cdot \boldsymbol{\Sigma} \cdot \mathbf{e}, \quad (2.19)$$

or the current field,

$$W_j(\mathbf{j}) = \mathbf{j} \cdot \boldsymbol{\Sigma}^{-1} \cdot \mathbf{j}. \quad (2.20)$$

It will be convenient for us to characterize the electrical properties of the material by the sum of energies that are stored in it under the action of two orthogonal electrical fields $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$:

$$W_E = W_e(\mathbf{e}^{(1)}) + W_e(\mathbf{e}^{(2)}). \quad (2.21)$$

Such a functional reflects the properties of the medium in two linear independent directions, and therefore characterizes the whole conductivity tensor of any anisotropic composite, unlike the functionals (2.19) or (2.20) that depend only on the properties of the medium in a fixed direction of the applied field. We may treat this sum as a quadratic form of the matrix $\mathbf{E} = (\mathbf{e}^{(1)} \mathbf{e}^{(2)})$:

$$W_E(\mathbf{E}) = (\mathbf{e}^{(1)} \mathbf{e}^{(2)}) \begin{pmatrix} \boldsymbol{\Sigma} & 0 \\ 0 & \boldsymbol{\Sigma} \end{pmatrix} \begin{pmatrix} \mathbf{e}^{(1)} \\ \mathbf{e}^{(2)} \end{pmatrix}. \quad (2.22)$$

It is convenient to use the representation of this matrix in the basis similar to one that we used in the elasticity problem, namely, in any fixed basis \mathbf{i} , \mathbf{j} we can treat the pair $(\mathbf{e}^{(1)} \mathbf{e}^{(2)})$ as the (2×2) matrix,

$$(\mathbf{e}^{(1)} \mathbf{e}^{(2)}) = \begin{pmatrix} e_1^{(1)} & e_1^{(2)} \\ e_2^{(1)} & e_2^{(2)} \end{pmatrix}. \quad (2.23)$$

In the basis (2.2) of (2×2) matrices, the quadratic form (2.22) can be written as

$$W_E(\mathbf{E}) = \mathbf{E} \cdot \hat{\boldsymbol{\Sigma}} \cdot \mathbf{E} \\ = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}^T \begin{pmatrix} \frac{1}{2}(\lambda_1 + \lambda_2) & \frac{1}{2}(\lambda_1 - \lambda_2) & 0 & 0 \\ \frac{1}{2}(\lambda_1 - \lambda_2) & \frac{1}{2}(\lambda_1 + \lambda_2) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\lambda_1 + \lambda_2) & \frac{1}{2}(\lambda_1 - \lambda_2) \\ 0 & 0 & \frac{1}{2}(\lambda_1 - \lambda_2) & \frac{1}{2}(\lambda_1 + \lambda_2) \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}, \quad (2.24)$$

where

$$\left. \begin{aligned} E_1 = \mathbf{E} : \mathbf{a}_1 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1^{(1)} + \mathbf{e}_2^{(2)}), & E_2 = \mathbf{E} : \mathbf{a}_2 &= \frac{1}{\sqrt{2}}(\mathbf{e}_1^{(1)} - \mathbf{e}_2^{(2)}), \\ E_3 = \mathbf{E} : \mathbf{a}_3 &= \frac{1}{\sqrt{2}}(\mathbf{e}_2^{(1)} + \mathbf{e}_1^{(2)}), & E_4 = \mathbf{E} : \mathbf{a}_4 &= \frac{1}{\sqrt{2}}(\mathbf{e}_2^{(1)} - \mathbf{e}_1^{(2)}), \end{aligned} \right\} \quad (2.25)$$

and λ_1 and λ_2 are the principal conductivities of the material whose eigenvectors coincide with the directions of the vectors \mathbf{i} and \mathbf{j} in (2.2). Henceforth, we use the

schematic notation $\mathbf{A} \cdot \mathbf{B}$ for the matrix product of matrices \mathbf{A} and \mathbf{B} as in (2.24). Relation (2.24) define the matrix $\hat{\Sigma}$ that we use later. For an isotropic material, the matrix $\hat{\Sigma}$ in the basis (2.2) has a diagonal form,

$$\hat{\Sigma} = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \sigma \end{pmatrix}, \quad (2.26)$$

where σ is the conductivity constant of this isotropic material.

Similarly, the sum of the energies stored by conducting material in a current fields $\mathbf{j}^{(1)}$ and $\mathbf{j}^{(2)}$ can be written in the quadratic form

$$W_J = \mathbf{J} \cdot \hat{\Sigma}^{-1} \cdot \mathbf{J} \\ = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix}^T \begin{pmatrix} \frac{1}{2}(\lambda_1^{-1} + \lambda_2^{-1}) & \frac{1}{2}(\lambda_1^{-1} - \lambda_2^{-1}) & 0 & 0 \\ \frac{1}{2}(\lambda_1^{-1} - \lambda_2^{-1}) & \frac{1}{2}(\lambda_1^{-1} + \lambda_2^{-1}) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\lambda_1^{-1} + \lambda_2^{-1}) & \frac{1}{2}(\lambda_1^{-1} - \lambda_2^{-1}) \\ 0 & 0 & \frac{1}{2}(\lambda_1^{-1} - \lambda_2^{-1}) & \frac{1}{2}(\lambda_1^{-1} + \lambda_2^{-1}) \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix}, \quad (2.27)$$

where

$$\mathbf{J} = (\mathbf{j}^{(1)} \mathbf{j}^{(2)}), \quad J_i = \mathbf{J} : \mathbf{a}_i, \quad i = 1, 2, 3, 4. \quad (2.28)$$

(b) Homogenization

Let us consider a composite that is a double-periodic plane structure. The element of periodicity Ω is divided into two parts Ω_1 and Ω_2 with volume fractions f_1 and $f_2 = 1 - f_1$, respectively. Let us assume that these two parts are occupied by two isotropic materials with the elastic moduli (κ_1, μ_1) and (κ_2, μ_2) , and with the electrical conductivities σ_1 and σ_2 . It is desired to study the homogenization problem, i.e. the problem of describing the medium's effective properties. It is well known (Beran 1968; Christensen 1979; Sanchez-Palencia 1980) that the average behaviour of a mixture is described by the homogenized equations of elasticity,

$$\langle \boldsymbol{\epsilon} \rangle = \frac{1}{2} \langle \nabla \langle \mathbf{u} \rangle + (\nabla \langle \mathbf{u} \rangle)^T \rangle, \quad \langle \boldsymbol{\tau} \rangle = \langle \boldsymbol{\tau} \rangle^T, \quad \langle \boldsymbol{\tau} \rangle = \mathbf{C}_* : \langle \boldsymbol{\epsilon} \rangle, \quad \nabla \cdot \langle \boldsymbol{\tau} \rangle = 0, \quad (2.29)$$

and of conductivity

$$\langle \mathbf{j} \rangle = -\nabla \langle \phi \rangle, \quad \langle \mathbf{j} \rangle = \boldsymbol{\Sigma}_* \cdot \langle \mathbf{e} \rangle, \quad \nabla \cdot \langle \mathbf{j} \rangle = 0. \quad (2.30)$$

Here the symbol $\langle (\cdot) \rangle$ denotes averaging over the element of periodicity Ω , i.e.

$$\langle (\cdot) \rangle = \frac{1}{\text{vol } \Omega} \int_{\Omega} (\cdot) \, d\Omega. \quad (2.31)$$

The tensor \mathbf{C}_* , connecting the average stress and average strain, is by definition the *effective stiffness tensor*, and the tensor $\boldsymbol{\Sigma}_*$, connecting the average current and average electrical field is the *effective conductivity tensor*. The effective property tensors \mathbf{C}_* and $\boldsymbol{\Sigma}_*$ depend on the phase properties, phase volume fractions f_1 and f_2 , and the geometrical structure of the composite, independent of the loading.

Remark 2.4. Note that any homogeneous composite is equivalent in respect to the effective elasticity and conductivity tensors to some periodic structure. The assumption of periodicity is not very restrictive; it is imposed only for the sake of simplicity of description.

The elastic energy density W^* stored in the composite is known to be equal to

$$W_{\epsilon}^*(\epsilon_0) = \epsilon_0 : \mathbf{C}_* : \epsilon_0 = \inf_{\substack{\epsilon : \langle \epsilon \rangle = \epsilon_0 \\ \epsilon = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2}} \langle \epsilon : \mathbf{C} : \epsilon \rangle, \quad (2.32)$$

where infimum is taken over all admissible fields $\epsilon = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$ that are the strain fields with given mean value ϵ_0 that satisfy compatibility conditions (Atkin & Fox 1990; Willis 1981). For the conjugate functional of the complementary energy (Atkin & Fox 1990; Willis 1981) we have

$$W_{\tau}^*(\tau) = \tau_0 : \mathbf{S}_* : \tau_0 = \inf_{\substack{\tau : \langle \tau \rangle = \tau_0 \\ \tau = \tau^T, \quad \nabla \cdot \tau = 0}} \langle \tau : \mathbf{S} : \tau \rangle, \quad (2.33)$$

where the effective compliance tensor \mathbf{S}_* is determined as $\mathbf{S}_* = \mathbf{C}_*^{-1}$ and infimum is taken over stress fields with given mean value τ_0 that satisfy equilibrium conditions $\nabla \cdot \tau = 0$.

The electrostatic energy density of the composite is known to be a quadratic form in the electrical field, i.e.

$$W_e^*(e_0) = e_0 \cdot \Sigma_* \cdot e_0 = \inf_{\substack{e : \langle e \rangle = e_0 \\ e = -\nabla \phi}} \langle e \cdot \Sigma \cdot e \rangle \quad (2.34)$$

(Dirichlet variational principle (see Beran 1968)), or the current field,

$$W_j^*(j_0) = j_0 \cdot \Sigma_*^{-1} \cdot j_0 = \inf_{\substack{j : \langle j \rangle = j_0 \\ \nabla \cdot j = 0}} \langle j \cdot \Sigma^{-1} \cdot j \rangle \quad (2.35)$$

(Thomson variational principle (see Beran 1968)). For the conductivity problem we use the functionals that are the sums of the energy stored by the composite in two trial fields, like (2.24) and (2.27), namely,

$$W_E^*(E_0) = E_0 \cdot \hat{\Sigma}_* \cdot E_0 = \inf_{\substack{E : \langle E \rangle = E_0 \\ E = -\nabla(\phi^{(1)}, \phi^{(2)})}} \langle E \cdot \hat{\Sigma} \cdot E \rangle, \quad (2.36)$$

and

$$W_J^*(J_0) = J_0 \cdot \hat{\Sigma}_*^{-1} \cdot J_0 = \inf_{\substack{J : \langle J \rangle = J_0 \\ \nabla \cdot J = 0}} \langle J \cdot \hat{\Sigma}^{-1} \cdot J \rangle, \quad (2.37)$$

where the tensor $\hat{\Sigma}_*$ is defined similar to (2.24).

3. The translation method

To prove our cross-property bounds we will use the translation method (see Lurie & Cherkaev 1984*a, b*, 1986*a, b*; Murat & Tartar 1985; Tartar 1985; Kohn & Strang 1986; Cherkaev & Gibiansky 1984, 1987, 1992, 1993; Francfort & Murat 1986; Milton 1990*a, b*; Gibiansky & Milton 1993). The method is based on bounding from below the relevant energy functional I that for the problem under study has the general

form

$$I = \sum_{i=1}^N W_{\epsilon}^*(\epsilon_0^{(i)}) + W_E^*(\mathbf{E}_0) \quad (3.1)$$

(see relations (2.32)–(2.33), (2.36)–(2.37)). This functional is equal to the sum of the values of elastic and electrical energy stored in the element of periodicity of a composite which is exposed to N elastic fields and two electrical fields with fixed mean values. Here $N = 1$ for the bulk modulus bounds and $N = 2$ for the shear modulus bounds. Note that the energy can be treated also in terms of the stress and the current fields for the elastic and electrical parts of the functional (3.1), respectively. The energy functional is used because its value is equal to the energy stored by an equivalent homogeneous medium in the uniform field. The equivalent medium is characterized by the tensor of the effective properties and the uniform external field coincides with the mean value of the field in the composite. Clearly, the lower bound on the functional (3.1) provides bounds on the effective moduli of interest.

(a) Functionals

We discuss now the functionals that yield our cross-property bounds and specify the functionals of the type (3.1) which attain minimal values at the boundary of the set of pairs (σ_*, κ_*) and (σ_*, μ_*) . We follow here the prescription of Cherkaev & Gibiansky (1993).

It is instructive to begin with functionals for the pure elasticity and pure conductivity problems before presenting the cross-property bounds. To obtain bounds on the bulk modulus, the composite is exposed to an external hydrostatic strain $\epsilon_h = \epsilon_h \mathbf{a}_1$ or stress $\tau_h = \tau_h \mathbf{a}_1$ field. Indeed, the energy of an isotropic composite under the action of these fields is proportional to the effective bulk modulus κ_* according to

$$I_{\epsilon}(\epsilon_h) = W_{\epsilon}^*(\epsilon_h) = \epsilon_h : \mathbf{C}_* : \epsilon_h = 2\kappa_*(\epsilon_h)^2, \quad (3.2)$$

or to its inverse value $1/\kappa_*$ according to

$$I_{\tau}(\tau_h) = W_{\tau}^*(\tau_h) = \tau_h : \mathbf{S}_* : \tau_h = \frac{1}{2\kappa_*}(\tau_h)^2 \quad (3.3)$$

(see relations (2.9) and (2.12)). The minimization (by changing the microstructure) of the energy stored in the the hydrostatic strain field (i.e. the functional (3.2)) is equivalent to the minimization of the bulk modulus. By contrast the minimization of the energy stored in the hydrostatic stress field (see (3.3)) leads to maximization of this modulus. As we will see further, we need to express the elastic energy in terms of ζ (gradient of the displacement vector) instead of the strain fields ϵ , i.e.

$$I_{\epsilon}(\epsilon_h) = I_{\zeta}(\zeta_h) = \zeta_h : \mathbf{C}'_* : \zeta_h, \quad (3.4)$$

where we assume that $\epsilon_h : \mathbf{a}_i = \zeta_h : \mathbf{a}_i$, $i = 1, 2, 3$.

By using the same argument, it is clear if $\mathbf{E} = \mathbf{E}_h = E_h \mathbf{a}_1$, then the minimization of the functional

$$I_E(\mathbf{E}_h) = W_E^*(\mathbf{E}_h) = \mathbf{E}_h : \hat{\Sigma} : \mathbf{E}_h = \sigma_*(E_h)^2 \quad (3.5)$$

is equivalent to the minimization of the effective conductivity constant σ_* of an

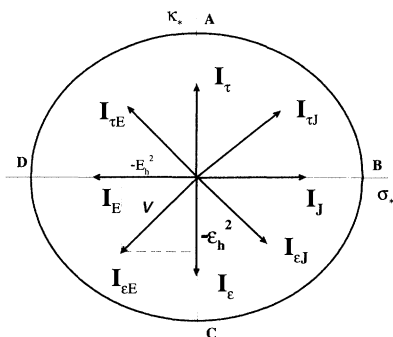


Figure 3. Directions opposite to the gradients of various functionals described in the text. For example, \mathbf{v} represents the direction opposite to the gradient of the functional I_{eE} ; $-E_h^2$ and $-\epsilon_h^2$ are the components of the vector \mathbf{v} in the σ_* and κ_* directions. The ellipse demarcates schematically the boundary of the set of the admissible pairs (σ_*, κ_*) .

isotropic composite. If $\mathbf{J} = \mathbf{J}_h = J_h \mathbf{a}_1$ then the minimization of the functional

$$I_J(\mathbf{J}_h) = W_J^*(\mathbf{J}_h) = \mathbf{J}_h : \hat{\Sigma}^{-1} : \mathbf{J}_h = \frac{1}{\sigma_*} (J_h)^2 \quad (3.6)$$

is equivalent to maximization of the effective conductivity constant.

In order to get conductivity-bulk modulus bounds, we combine the functionals mentioned above. We have a choice between stress and strain trial fields for the elasticity problem, and between the electrical and current fields for the electrostatic problem. The following functionals should be considered for the conductivity-bulk modulus bounds:

$$I_{\zeta E}(\zeta_h, \mathbf{E}_h) = W_\zeta^*(\zeta_h) + W_E^*(\mathbf{E}_h), \quad (3.7)$$

$$I_{\zeta J}(\zeta_h, \mathbf{J}_h) = W_\zeta^*(\zeta_h) + W_J^*(\mathbf{J}_h), \quad (3.8)$$

$$I_{\tau E}(\boldsymbol{\tau}_h, \mathbf{E}_h) = W_\tau^*(\boldsymbol{\tau}_h) + W_E^*(\mathbf{E}_h), \quad (3.9)$$

$$I_{\tau J}(\boldsymbol{\tau}_h, \mathbf{J}_h) = W_\tau^*(\boldsymbol{\tau}_h) + W_J^*(\mathbf{J}_h). \quad (3.10)$$

The lower bound of each of these functionals gives some component of the boundary. This is illustrated in the figure 3 where the directions opposite to the gradients of all of these functionals are shown in the conductivity-bulk modulus plane $\sigma_*-\kappa_*$. Let us assume that we start with some microstructure that possesses the effective properties (σ_*, κ_*) and we place the origin of the plane at this point. If one changes the microstructure with the aim to decrease the energy $W_\zeta^*(\zeta_h)$, one moves vertically down in this plane, corresponding to a lower κ_* . Decreasing the energy $W_\tau^*(\boldsymbol{\tau}_h)$ allows one to move vertically up in this plane, corresponding to the larger κ_* . Similarly, the minimization of the functional $W_E^*(\mathbf{E}_h)$ moves one horizontally to the left, corresponding to a lower σ_* ; minimization of the functional $W_J^*(\mathbf{J}_h)$ moves one horizontally to the right in this plane, corresponding to a larger σ_* .

By using linear combinations (3.7)–(3.10) of the four functionals (3.2), (3.3) and (3.5), (3.6), one can find the functional that moves one in any fixed direction in the $\sigma_*-\kappa_*$ plane. Therefore, by bounding the functionals (3.7)–(3.10) from below one can

find bounds on the set of pairs of the values (σ_*, κ_*) of composites of all possible microstructures. For example, consider the functional (3.7). For any fixed amplitudes E_h and ϵ_h , the bounds on the functional (3.7) show how far one can move in the direction $\mathbf{v} = -E_h^2 \mathbf{i} - \epsilon_h^2 \mathbf{j}$ where \mathbf{i} and \mathbf{j} are the unit vectors of the σ_* -axis and κ_* -axis, respectively (see figure 3). By changing the ratio E_h^2/ϵ_h^2 one can direct \mathbf{v} at any point within the third quadrant of the $\sigma_*-\kappa_*$ plane. This means that the functional (3.7) can provide the part of the cross-property boundary having a normal directed into the third quadrant (like (CD) in figure 3). Similarly, the functional (3.8) provides the part of the boundary having a normal directed into the second quadrant of the plane $\sigma_*-\kappa_*$ (like (DA) in figure 3), the functional (3.9) corresponds to the part of the boundary having a normal directed into the fourth quadrant of the plane (σ_*, κ_*) (like (BC) in figure 3), and the functional (3.10) provides the part of the boundary having a normal directed into the first quadrant of the plane (σ_*, κ_*) (like (AB) in figure 3).

In a similar manner, one can get bounds on the shear modulus of a composite by bounding the energy stored in a composite exposed to shear-type trial strain or stress fields $\epsilon_s = \epsilon_s \mathbf{a}_1$ or $\tau_s = \tau_s \mathbf{a}_1$, i.e.

$$I_\epsilon(\epsilon_s) = W_\epsilon^*(\epsilon_s) = \epsilon_s : \mathbf{C}_* : \epsilon_s, \quad (3.11)$$

or

$$I_\tau(\tau_s) = W_\tau^*(\tau_s) = \tau_s : \mathbf{S}_* : \tau_s. \quad (3.12)$$

In this way we obtain bounds on any of the two shear moduli of the mixture which is anisotropic in general. To ensure isotropy of the mixture, one need consider the reaction of the composite to two orthogonal shear fields. Hence to estimate the shear modulus of an isotropic composite we should minimize the functional equal to the sum of the values of the energy stored by the medium under the action of two trial orthogonal shear strain or stress fields $\epsilon_s^{(1)} = \epsilon_s^{(1)} \mathbf{a}_2$, $\epsilon_s^{(2)} = \epsilon_s^{(2)} \mathbf{a}_3$, or $\tau_s^{(1)} = \tau_s^{(1)} \mathbf{a}_2$, $\tau_s^{(2)} = \tau_s^{(2)} \mathbf{a}_3$, i.e.

$$\begin{aligned} I_{\epsilon\epsilon}(\epsilon_s^{(1)}, \epsilon_s^{(2)}) &= W_\epsilon^*(\epsilon_s^{(1)}) + W_\epsilon^*(\epsilon_s^{(2)}) \\ &= \epsilon_s^{(1)} : \mathbf{C}_* : \epsilon_s^{(1)} + \epsilon_s^{(2)} : \mathbf{C}_* : \epsilon_s^{(2)} = 2\mu_*((\epsilon_s^{(1)})^2 + (\epsilon_s^{(2)})^2) \end{aligned} \quad (3.13)$$

(see (2.9)), or

$$\begin{aligned} I_{\tau\tau}(\tau_s^{(1)}, \tau_s^{(2)}) &= W_\tau^*(\tau_s^{(1)}) + W_\tau^*(\tau_s^{(2)}) \\ &= \tau_s^{(1)} : \mathbf{S}_* : \tau_s^{(1)} + \tau_s^{(2)} : \mathbf{S}_* : \tau_s^{(2)} = \frac{1}{2\mu_*}((\tau_s^{(1)})^2 + (\tau_s^{(2)})^2) \end{aligned} \quad (3.14)$$

(see (2.12)). The functional (3.13) can be reexpressed in terms of ζ as in the relation (3.4), i.e.

$$I_{\epsilon\epsilon}(\epsilon_s^{(1)}, \epsilon_s^{(2)}) = I_{\zeta\zeta}(\zeta_s^{(1)}, \zeta_s^{(2)}) = \zeta_s^{(1)} : \mathbf{C}'_* : \zeta_s^{(1)} + \zeta_s^{(2)} : \mathbf{C}'_* : \zeta_s^{(2)} = 2\mu_*((\zeta_s^{(1)})^2 + (\zeta_s^{(2)})^2), \quad (3.15)$$

where $\epsilon_s^{(1)} : \mathbf{a}_i = \zeta_s^{(1)} : \mathbf{a}_i$, and $\epsilon_s^{(2)} : \mathbf{a}_i = \zeta_s^{(2)} : \mathbf{a}_i$, $i = 1, 2, 3$.

Hence, in order to find conductivity-shear modulus bounds, we study the following functionals:

$$I_{\zeta\zeta E}(\zeta_s^{(1)}, \zeta_s^{(2)}, \mathbf{E}_h) = W_\zeta^*(\zeta_s^{(1)}) + W_\zeta^*(\zeta_s^{(2)}) + W_E^*(\mathbf{E}_h), \quad (3.16)$$

$$I_{\zeta\zeta J}(\zeta_s^{(1)}, \zeta_s^{(2)}, \mathbf{J}_h) = W_\zeta^*(\zeta_s^{(1)}) + W_\zeta^*(\zeta_s^{(2)}) + W_J^*(\mathbf{J}_h), \quad (3.17)$$

$$I_{\tau\tau E}(\boldsymbol{\tau}_s^{(1)}, \boldsymbol{\tau}_s^{(2)}, \mathbf{E}_h) = W_\tau^*(\boldsymbol{\tau}_s^{(1)}) + W_\tau^*(\boldsymbol{\tau}_s^{(2)}) + W_E^*(\mathbf{E}_h), \quad (3.18)$$

$$I_{\tau\tau J}(\boldsymbol{\tau}_s^{(1)}, \boldsymbol{\tau}_s^{(2)}, \mathbf{J}_h) = W_\tau^*(\boldsymbol{\tau}_s^{(1)}) + W_\tau^*(\boldsymbol{\tau}_s^{(2)}) + W_J^*(\mathbf{J}_h). \quad (3.19)$$

The lower bound on each of these functionals gives some component of the boundary as was explained above for the conductivity-bulk modulus bounds.

(b) *Translation bounds*

First, note that each of the functionals described above is a quadratic form of the elastic and electrical fields and can be represented in the form,

$$I = \mathbf{e}_0 \cdot \mathbf{D}_* \cdot \mathbf{e}_0 = \inf_{\substack{\mathbf{e}: \langle \mathbf{e} \rangle = \mathbf{e}_0, \\ \mathbf{e} \in EK}} \langle \mathbf{e} \cdot \mathbf{D}(\mathbf{x}) \cdot \mathbf{e} \rangle, \quad (3.20)$$

where infimum is taken over fields \mathbf{e} with given mean value \mathbf{e}_0 such that

$$\mathbf{e} \in EK. \quad (3.21)$$

Here \mathbf{e} is a vector composed of the coefficients of tensors of gradients $\boldsymbol{\zeta}$ or stresses $\boldsymbol{\tau}$ and matrices \mathbf{E} or \mathbf{J} in the basis (2.2). The set EK is a set of doubly periodic vectors that satisfy some differential restrictions. For the components of stress tensor, these restrictions are given by the equilibrium equations $\nabla \cdot \boldsymbol{\tau} = 0$, while for gradients one has $\boldsymbol{\zeta} = \nabla \mathbf{u}$. For the matrix $\mathbf{E} = \nabla(\phi_1, \phi_2)$ of the electrical fields, these restrictions guarantee the potential character of these fields, and for the pair of current fields \mathbf{J} they are given by the conditions $\nabla \cdot \mathbf{J} = 0$. The matrix \mathbf{D} is a piecewise constant block diagonal matrix composed of the coefficients of the material tensors in the aforementioned basis (2.2).

For example, the matrix \mathbf{D} , the vector \mathbf{e} , and the set EK for the functional $I_{\zeta E}(\boldsymbol{\zeta}, \mathbf{E})$ have the form,

$$\mathbf{D} = \mathbf{D}^{\zeta E} = \begin{pmatrix} \mathbf{C}' & 0 \\ 0 & \hat{\boldsymbol{\Sigma}} \end{pmatrix}, \quad (3.22)$$

$$\mathbf{e} = [\boldsymbol{\zeta}, \mathbf{E}] = \left\{ \begin{aligned} & [(\boldsymbol{\zeta} : \mathbf{a}_1, \boldsymbol{\zeta} : \mathbf{a}_2, \boldsymbol{\zeta} : \mathbf{a}_3, \boldsymbol{\zeta} : \mathbf{a}_4), (\mathbf{E} : \mathbf{a}_1, \mathbf{E} : \mathbf{a}_2, \mathbf{E} : \mathbf{a}_3, \mathbf{E} : \mathbf{a}_4)], \\ & EK = \{ \mathbf{e} : \boldsymbol{\zeta} = \nabla \mathbf{u}, \quad \mathbf{E} = \nabla(\phi_1, \phi_2) \}, \end{aligned} \right\} \quad (3.23)$$

where the matrices \mathbf{C}' and $\hat{\boldsymbol{\Sigma}}$ are given by relations (2.16) and (2.24), respectively.

The translation method allows one to take into account the integral corollaries of the linear differential restrictions (3.21). Namely, given (3.21), it is possible to find so called *quasi-convex* (see Dacorogna (1982) and references therein) quadratic functions of the fields \mathbf{e}

$$\phi(\mathbf{e}) = \mathbf{e} \cdot \mathbf{T} \cdot \mathbf{e} \quad (3.24)$$

possessing the property of convex functions

$$\langle \phi(\mathbf{e}) \rangle \geq \phi(\langle \mathbf{e} \rangle) \quad (3.25)$$

for every field $\mathbf{e} \in EK$. Here \mathbf{T} is the so called translation matrix which is a constant matrix. Of course, any positive-definite matrix is a matrix of quasi-convex (even convex) quadratic form. However, under the restrictions (3.21), it is possible to find non-positively defined matrices \mathbf{T} such that the associated quadratic form possesses the quasi-convexity property (3.25). Explicit forms of these matrices depend on the

differential restrictions (3.21). These matrices can depend on several free parameters. If the inequality in the expression (3.25) is valid as an equality for any field $\mathbf{e} \in EK$, such a function behaves as an affine function and is called *quasi-affine*. For the problem under study (i.e. for the stress and strain fields and gradient of vector potential), such quasi-affine bilinear and quadratic functions were studied in detail by Cherkaev & Gibiansky (1993). Let us now describe how to use these quasi-affine function in order to get sharp bounds on the effective properties. We follow here the description of Milton (1990b).

Consider a two-phase composite with the local constitutive relation

$$\mathbf{j}(\mathbf{x}) = \mathbf{D}(\mathbf{x}) \cdot \mathbf{e}(\mathbf{x})$$

at a point \mathbf{x} . Here \mathbf{j} is a generalized ‘flux’, \mathbf{e} is a generalized ‘gradient’, and \mathbf{D} is some local property, generally a tensor, equal to \mathbf{D}_1 in phase 1 and \mathbf{D}_2 in phase 2. For example, in the *pure* conduction (elasticity) problem, \mathbf{j} , \mathbf{e} and \mathbf{D} represent the current (stress), electric field (strain), and conductivity tensor (stiffness tensor), respectively. In the present problem, \mathbf{D} is actually a ‘supertensor’ (e.g. see (3.22)) as discussed below. The effective tensor \mathbf{D}_* is defined by the relation,

$$\langle \mathbf{j} \rangle = \mathbf{D}_* \cdot \langle \mathbf{e} \rangle,$$

or equivalently by the averaged energy expression

$$\langle \mathbf{e} \cdot \mathbf{D} \cdot \mathbf{e} \rangle = \langle \mathbf{e} \rangle \cdot \mathbf{D}_* \cdot \langle \mathbf{e} \rangle,$$

where \mathbf{e} and \mathbf{j} is the solution of the corresponding system of differential equations for the medium with the properties $\mathbf{D}(\mathbf{x})$. It can be alternatively defined by the variational principle

$$\mathbf{e}_0 \cdot \mathbf{D}_* \cdot \mathbf{e}_0 = \inf_{\substack{\mathbf{e}: \langle \mathbf{e} \rangle = \mathbf{e}_0, \\ \mathbf{e} \in EK}} \langle \mathbf{e} \cdot \mathbf{D}(\mathbf{x}) \cdot \mathbf{e} \rangle. \quad (3.26)$$

Now consider a ‘comparison’ medium with local property tensor

$$\mathbf{D}'(\mathbf{x}) = \mathbf{D}(\mathbf{x}) - \mathbf{T}, \quad (3.27)$$

where \mathbf{T} is a constant *translation* tensor chosen in such a way that (i) \mathbf{D}' is positive semi-definite and (ii) the quadratic form associated with \mathbf{T} is *quasi-convex*.

The effective properties of such medium can be defined via

$$\mathbf{e}_0 \cdot \mathbf{D}'_* \cdot \mathbf{e}_0 = \inf_{\substack{\mathbf{e}: \langle \mathbf{e} \rangle = \mathbf{e}_0, \\ \mathbf{e} \in EK}} \langle \mathbf{e} \cdot (\mathbf{D}(\mathbf{x}) - \mathbf{T}) \cdot \mathbf{e} \rangle \quad (3.28)$$

(cf. with (3.26)). Let $\mathbf{e}'(\mathbf{x})$ be a solution of the variational problem (3.26) and let us use this field as a trial field for the variational problem (3.28). This yields

$$\inf_{\substack{\mathbf{e}: \langle \mathbf{e} \rangle = \mathbf{e}_0, \\ \mathbf{e} \in EK}} \langle \mathbf{e} \cdot (\mathbf{D}(\mathbf{x}) - \mathbf{T}) \cdot \mathbf{e} \rangle \leq \langle \mathbf{e}' \cdot \mathbf{D}(\mathbf{x}) \cdot \mathbf{e}' \rangle - \langle \mathbf{e}' \cdot \mathbf{T} \cdot \mathbf{e}' \rangle \leq \mathbf{e}_0 \cdot \mathbf{D}_* \cdot \mathbf{e}_0 - \mathbf{e}_0 \cdot \mathbf{T} \cdot \mathbf{e}_0, \quad (3.29)$$

where we took into account of the quasi-convexity of the quadratic form with the matrix \mathbf{T} and the equation (3.26) that is an equality for the field $\mathbf{e} = \mathbf{e}'$. Hence, the effective properties of the comparison and original media are related by

$$\mathbf{D}_* - \mathbf{T} \geq \mathbf{D}'_* \quad (3.30)$$

Now use of the well-known harmonic mean bound (Christensen 1979) yields

$$(\mathbf{D}_* - \mathbf{T}) \geq \mathbf{D}'_* \geq [f_1(\mathbf{D}_1 - \mathbf{T})^{-1} + f_2(\mathbf{D}_2 - \mathbf{T})^{-1}]^{-1}, \quad (3.31)$$

or

$$(\mathbf{D}_* - \mathbf{T})^{-1} \leq f_1(\mathbf{D}_1 - \mathbf{T})^{-1} + f_2(\mathbf{D}_2 - \mathbf{T})^{-1}, \quad (3.32)$$

that is true for any matrix \mathbf{T} of a quasi-convex quadratic form such that

$$\mathbf{D}(\mathbf{x}) - \mathbf{T} \geq 0 \quad \text{for any } \mathbf{x}. \quad (3.33)$$

For two-phase composites, the restriction (3.33) means

$$\mathbf{D}_1 - \mathbf{T} \geq 0, \quad \mathbf{D}_2 - \mathbf{T} \geq 0. \quad (3.34)$$

Equation (3.32) is the basic inequality of the translation method. In fact, it is valid for any number of phases when written in a form

$$(\mathbf{D}_* - \mathbf{T})^{-1} \leq \langle (\mathbf{D}(\mathbf{x}) - \mathbf{T})^{-1} \rangle, \quad (3.35)$$

where \mathbf{T} is subjected to (3.33). The essential point is that one wants to choose \mathbf{T} so as to optimize the bound, to make it as restrictive as possible for the effective property tensor \mathbf{D}_* .

One can transform the bounds by using the so-called Y -transformation (Milton 1991; Cherkaev & Gibiansky 1992)

$$\mathbf{Y}(\mathbf{D}_1, \mathbf{D}_2, f_1, \mathbf{D}_*) = -f_2\mathbf{D}_1 - f_1\mathbf{D}_2 - f_1f_2(\mathbf{D}_1 - \mathbf{D}_2) \cdot (\mathbf{D}_* - f_1\mathbf{D}_1 - f_2\mathbf{D}_2)^{-1} \cdot (\mathbf{D}_1 - \mathbf{D}_2). \quad (3.36)$$

Henceforth, we will omit the first three arguments of the Y -transformation and will denote it simply as $\mathbf{Y}(\mathbf{D}_*)$. If the matrix $(\mathbf{D}_1 - \mathbf{D}_2)$ is not degenerate, then the bound (3.32) can be presented in a surprisingly simple form (Milton 1991; Cherkaev & Gibiansky 1992)

$$\mathbf{Y}(\mathbf{D}_*) + \mathbf{T} \geq 0. \quad (3.37)$$

Note that the bounds (3.37) in terms of the Y -transformations do not depend on the volume fraction. All volume-fraction information is 'hidden' in the definition of the Y -transformation.

Remark 3.1. Note also some of the remarkable properties of this transformation

$$\left. \begin{aligned} \mathbf{Y}(\mathbf{D}_1, \mathbf{D}_2, f_1, \mathbf{D}_i) &= -\mathbf{D}_i, \quad i = 1, 2, \\ \mathbf{Y}(\mathbf{D}_1^{-1}, \mathbf{D}_2^{-1}, f_1, \mathbf{D}_*^{-1}) &= \mathbf{Y}^{-1}(\mathbf{D}_1, \mathbf{D}_2, f_1, \mathbf{D}_*), \end{aligned} \right\} \quad (3.38)$$

that we will use in order to transform the bounds.

We use the scalar corollary of the matrix inequality (3.37), namely,

$$\det[\mathbf{Y}(\mathbf{D}_*) + \mathbf{T}] \geq 0. \quad (3.39)$$

The parameters of the symmetric matrix \mathbf{T} should be chosen in order to make the bounds (3.39) the most restrictive ones.

In the problem under study, the matrix $\mathbf{D}_1 - \mathbf{D}_2$ may be degenerate, i.e. some of the eigenvectors and eigenvalues of the matrices \mathbf{D}_1 and \mathbf{D}_2 coincide. Indeed, for any material, the stiffness matrix \mathbf{C}' has one eigenvalue equal to zero (see (2.16)). The matrix \mathbf{C}' is in turn the diagonal block of matrices \mathbf{D} used in the functionals (3.7)–(3.8), and (3.16)–(3.17). One can find the appropriate form of the bounds (3.32) for this case as well (Cherkaev & Gibiansky 1993) but here we will not go into details.

As we will see, in our problem all of the matrices in the matrix inequality (3.37) are block-diagonal if the component materials and the composite are isotropic which is the case. For the block of this matrix that gives the bounds the differences $\mathbf{D}_1 - \mathbf{D}_2$ is not degenerate and we can use the bound in the form (3.37).

(c) *Quasi-affine functions*

In order to use the translation method to obtain cross-property bounds, we need to find the set of bilinear quasi-affine functions of vectors which possess elements that are the components of various combinations of stress, strain, current or electrical fields. We follow here the procedure described by Cherkaev & Gibiansky (1993). The details of the procedure for the problem at hand are given there, and hence we only summarize the results below.

Specifically, one can show that for any choice of the parameters t_1, t_2, t_3, t_4 :

(i) The functions

$$\phi_{\zeta\zeta}(\boldsymbol{\zeta}^{(1)}, \boldsymbol{\zeta}^{(2)}) = \boldsymbol{\zeta}^{(1)} : \boldsymbol{\Phi}_{\zeta\zeta}(t_1, t_2, t_3, t_4) : \boldsymbol{\zeta}^{(2)}, \quad (3.40)$$

$$\phi_{EE}(\mathbf{E}^{(1)}, \mathbf{E}^{(2)}) = \mathbf{E}^{(1)} : \boldsymbol{\Phi}_{\zeta\zeta}(t_1, t_2, t_3, t_4) : \mathbf{E}^{(2)}, \quad (3.41)$$

$$\phi_{E\zeta}(\mathbf{E}, \boldsymbol{\zeta}) = \mathbf{E} : \boldsymbol{\Phi}_{\zeta\zeta}(t_1, t_2, t_3, t_4) : \boldsymbol{\zeta}, \quad (3.42)$$

are quasi-affine if the fourth-order tensor $\boldsymbol{\Phi}_{\zeta\zeta}$ is represented in the basis $\mathbf{a}_1, \dots, \mathbf{a}_4$ (see (2.2)) as follows:

$$\boldsymbol{\Phi}_{\zeta\zeta}(t_1, t_2, t_3, t_4) = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ -t_2 & -t_1 & t_4 & t_3 \\ -t_3 & -t_4 & -t_1 & -t_2 \\ -t_4 & -t_3 & t_2 & t_1 \end{pmatrix}. \quad (3.43)$$

(ii) The function

$$\phi_{JJ}(\mathbf{J}^{(1)}, \mathbf{J}^{(2)}) = \mathbf{J}^{(1)} : \boldsymbol{\Phi}_{JJ}(t_1, t_2, t_3, t_4) : \mathbf{J}^{(2)} \quad (3.44)$$

is quasi-affine if the tensor $\boldsymbol{\Phi}_{JJ}$ of the quadratic form $\phi_{JJ}(\mathbf{J}^{(1)}, \mathbf{J}^{(1)})$ is given by

$$\boldsymbol{\Phi}_{JJ}(t_1, t_2, t_3, t_4) = \begin{pmatrix} t_1 & -t_2 & -t_3 & t_4 \\ t_2 & -t_1 & t_4 & -t_3 \\ t_3 & -t_4 & -t_1 & t_2 \\ -t_4 & t_3 & -t_2 & t_1 \end{pmatrix}. \quad (3.45)$$

(iii) The functions

$$\phi_{J\zeta}(\mathbf{J}, \boldsymbol{\zeta}) = \mathbf{J} : \boldsymbol{\Phi}_{J\zeta}(t_1, t_2, t_3, t_4) : \boldsymbol{\zeta} \quad (3.46)$$

and

$$\phi_{JE}(\mathbf{J}, \mathbf{E}) = \mathbf{J} : \boldsymbol{\Phi}_{J\zeta}(t_1, t_2, t_3, t_4) : \mathbf{E} \quad (3.47)$$

are quasi-affine if

$$\boldsymbol{\Phi}_{J\zeta}(t_1, t_2, t_3, t_4) = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ t_2 & t_1 & -t_4 & -t_3 \\ t_3 & t_4 & t_1 & t_2 \\ -t_4 & -t_3 & t_2 & t_1 \end{pmatrix}. \quad (3.48)$$

(iv) The function

$$\phi_{\tau\tau}(\boldsymbol{\tau}^{(1)}, \boldsymbol{\tau}^{(2)}) = \boldsymbol{\tau}^{(1)} : \boldsymbol{\Phi}_{\tau\tau}(t_1, t_2, t_3) : \boldsymbol{\tau}^{(2)} \quad (3.49)$$

is quasi-affine if

$$\Phi_{\tau\tau}(t_1, t_2, t_3) = \begin{pmatrix} t_1 & -t_2 & -t_3 \\ t_2 & -t_1 & t_4 \\ t_3 & -t_4 & -t_1 \end{pmatrix}. \quad (3.50)$$

(v) The function

$$\phi_{\tau J}(\boldsymbol{\tau}, \mathbf{J}) = \boldsymbol{\tau} : \Phi_{\tau J}(t_1, t_2, t_3) : \mathbf{J} \quad (3.51)$$

is quasi-affine if

$$\Phi_{\tau J}(t_1, t_2, t_3, t_4) = \begin{pmatrix} t_1 & -t_2 & -t_3 & t_4 \\ t_2 & -t_1 & t_4 & -t_3 \\ t_3 & -t_4 & -t_1 & t_2 \end{pmatrix}. \quad (3.52)$$

(vi) The function

$$\phi_{\tau E} = \boldsymbol{\tau} : \Phi_{\tau E}(t_1, t_2, t_3) : \mathbf{E} \quad (3.53)$$

is quasi-affine if

$$\Phi_{\tau E}(t_1, t_2, t_3, t_4) = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ t_2 & t_1 & -t_4 & -t_3 \\ t_3 & t_4 & t_1 & t_2 \end{pmatrix}. \quad (3.54)$$

Hence, we have all of the necessary quasi-affine combinations. In §§ 4 and 5 we use them in order to prove the bounds of § 1.

4. Coupled conductivity-bulk modulus bounds

Here we prove the conductivity-bulk modulus bounds stated in § 1 (statement 1.1) by using the aforementioned translation method. We describe in detail the bounds on the functional $I_{\epsilon E}(\boldsymbol{\epsilon}_h, \mathbf{E}_h)$ and the corresponding bounds on the effective properties of the composite. The other functionals can be analysed in a very similar way and hence we omit these details of the proof. As will be seen, we will need to prove the bounds separately for two cases depending on the sign of the expression $(\sigma_1 - \sigma_2)(\mu_1 - \mu_2)$. In this section we call the pair of the materials ‘well-ordered’ if

$$(\sigma_1 - \sigma_2)(\mu_1 - \mu_2) \geq 0, \quad (4.1)$$

in contrast to ‘badly ordered’ materials that satisfy the relation

$$(\sigma_1 - \sigma_2)(\mu_1 - \mu_2) \leq 0. \quad (4.2)$$

These definitions should not be confused with the commonly used ones that involve the bulk and shear moduli, and with ones used in § 5 that involve the conductivity and bulk modulus.

(a) Badly ordered materials

Let us assume that the phase properties satisfy relation (4.2) and begin with analysis of the bounds on the functional $I_{\epsilon E}(\boldsymbol{\epsilon}_h, \mathbf{E}_h)$. As mentioned earlier, it can be written in the quadratic form

$$I_{\epsilon E}(\boldsymbol{\epsilon}_h, \mathbf{E}_h) = \boldsymbol{\epsilon}_h : \mathbf{C}_* : \boldsymbol{\epsilon}_h + \mathbf{E}_h : \hat{\boldsymbol{\Sigma}}_* : \mathbf{E}_h = \boldsymbol{\zeta}_h : \mathbf{C}'_* : \boldsymbol{\zeta}_h + \mathbf{E}_h : \hat{\boldsymbol{\Sigma}}_* : \mathbf{E}_h = \mathbf{e}_0 \cdot \mathbf{D}_*^{\epsilon E} \cdot \mathbf{e}_0, \quad (4.3)$$

where the vector \mathbf{e} and the matrix of ‘supertensor’ $\mathbf{D}_* = \mathbf{D}_*^{\epsilon E}$ are defined by the relations (3.22)–(3.23). The translation matrix $\mathbf{T}^{\epsilon E}$ can be represented in a block

form as

$$T^{\epsilon E} = \begin{pmatrix} \Phi_{\zeta\zeta}(-t_1, 0, 0, 0) & \Phi_{\zeta\zeta}(-t_3, 0, 0, 0) \\ \Phi_{\zeta\zeta}^T(-t_3, 0, 0, 0) & \Phi_{\zeta\zeta}(-t_2, 0, 0, 0) \end{pmatrix}. \quad (4.4)$$

Here we take many of the parameters to be equal to zero. One can check that additional free parameters can not improve the bound. As follows from the representation (3.23) for the vector \mathbf{e} and from the relations (3.40)–(3.42), the quadratic form,

$$\psi_{\epsilon E} = \mathbf{e} \cdot \mathbf{T}^{\epsilon E} \cdot \mathbf{e} \quad (4.5)$$

is quasi-affine for any choice of the parameters t_1, t_2, t_3 . Therefore the bound can be written in the form

$$\mathbf{Y}(\mathbf{D}_*^{\epsilon E}) + \mathbf{T}^{\epsilon E} \geq 0 \quad (4.6)$$

where the parameters t_1, t_2, t_3 satisfy the relations

$$\begin{aligned} & \mathbf{D}_i^{\epsilon E} - \mathbf{T}^{\epsilon E} \\ &= \begin{pmatrix} 2\kappa_i + t_1 & 0 & 0 & 0 & t_3 & 0 & 0 & 0 \\ 0 & 2\mu_i - t_1 & 0 & 0 & 0 & -t_3 & 0 & 0 \\ 0 & 0 & 2\mu_i - t_1 & 0 & 0 & 0 & -t_3 & 0 \\ 0 & 0 & 0 & t_1 & 0 & 0 & 0 & t_3 \\ t_3 & 0 & 0 & 0 & \sigma_i + t_2 & 0 & 0 & 0 \\ 0 & -t_3 & 0 & 0 & 0 & \sigma_i - t_2 & 0 & 0 \\ 0 & 0 & -t_3 & 0 & 0 & 0 & \sigma_i - t_2 & 0 \\ 0 & 0 & 0 & -t_3 & 0 & 0 & 0 & \sigma_i + t_2 \end{pmatrix} \\ & \geq 0, \quad i = 1, 2. \end{aligned} \quad (4.7)$$

One can see that the matrix in the inequality (4.7) is a block-matrix composed of four (2×2) blocks, namely

$$\mathbf{D}_i^{\epsilon E} - \mathbf{T}^{\epsilon E} = \mathbf{A}_i^{1,5} \oplus \mathbf{A}_i^{2,6} \oplus \mathbf{A}_i^{3,7} \oplus \mathbf{A}_i^{4,8}. \quad (4.8)$$

Here $\mathbf{A}_i^{k,l}$ is a submatrix of the matrix $\mathbf{A}_i = \mathbf{D}_i^{\epsilon E} - \mathbf{T}^{\epsilon E}$ that is composed from the elements that are the intersections of the columns with numbers k and l and the rows with the same numbers, i.e.

$$\left. \begin{aligned} \mathbf{A}_i^{1,5} &= \begin{pmatrix} 2\kappa_i + t_1 & t_3 \\ t_3 & \sigma_i + t_2 \end{pmatrix}, & \mathbf{A}_i^{2,6} &= \mathbf{A}_i^{3,7} = \begin{pmatrix} 2\mu_i - t_1 & -t_3 \\ -t_3 & \sigma_i - t_2 \end{pmatrix}, \\ \mathbf{A}_i^{4,8} &= \begin{pmatrix} t_1 & t_3 \\ t_3 & \sigma_i + t_2 \end{pmatrix}. \end{aligned} \right\} \quad (4.9)$$

Obviously, each block is non-negative if the whole matrix is non-negative, and therefore we arrive at the following restrictions on the parameters t_1, t_2, t_3 :

$$\det \mathbf{A}_i^{1,5} = (2\kappa_i + t_1)(\sigma_i + t_2) - t_3^2 \geq 0, \quad i = 1, 2, \quad (4.10)$$

$$\det \mathbf{A}_i^{2,6} = \det \mathbf{A}_i^{3,7} = (2\mu_i - t_1)(\sigma_i - t_2) - t_3^2 \geq 0, \quad i = 1, 2, \quad (4.11)$$

$$\det \mathbf{A}_i^{4,8} = t_1(\sigma_i + t_2) - t_3^2 \geq 0, \quad i = 1, 2. \quad (4.12)$$

The left-hand side of the inequality (4.6) for any isotropic composite is also a block matrix; the bound that we seek comes from the first block of this matrix:

$$[\mathbf{Y}(\mathbf{D}_*^{\epsilon E}) - \mathbf{T}^{\epsilon E}]^{1,5} = \begin{pmatrix} 2y(\kappa_*) - t_1 & -t_3 \\ -t_3 & y(\sigma_*) - t_2 \end{pmatrix} \geq 0. \quad (4.13)$$

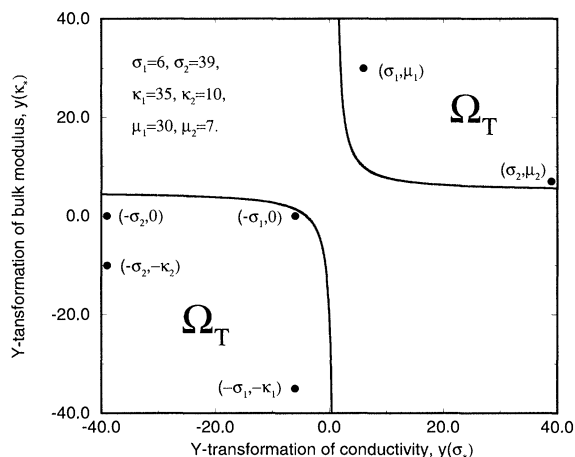


Figure 4. The set Ω_T in the $y(\sigma_*)$ - $y(\kappa_*)$ plane. This set contains the indicated points as described in the text.

Here $y(\kappa_*)$ is the Y -transformation of the effective bulk modulus, namely,

$$y(\kappa_*) = -f_1\kappa_2 - f_2\kappa_1 - \frac{f_1f_2(\kappa_1 - \kappa_2)^2}{\kappa_* - f_1\kappa_1 - f_2\kappa_2} \quad (4.14)$$

and $y(\sigma_*)$ is the Y -transformation of the effective conductivity. The determinant of the matrix of the left-hand side of (4.13) should be non-negative, and hence, we find the bound

$$\det[\mathbf{Y}(\mathbf{D}_*^{\epsilon E}) - \mathbf{T}^{\epsilon E}]^{1,5} = (2y(\kappa_*) - t_1)(y(\sigma_*) - t_2) - t_3^2 \geq 0. \quad (4.15)$$

Now we want to find the best choice of the parameters (subject to the restrictions (4.10)–(4.12)) in order to optimize the bound (4.15). We use here the geometrical interpretation of the translation bounds that was suggested by Gibiansky & Milton (1993). Let us denote as Ω_T the set of the pairs (σ_*, κ_*) that satisfy the relationship (4.15) for some matrix \mathbf{T} , i.e. for fixed values of the parameters (t_1, t_2, t_3) (figure 4).

The bounds of this set are defined by the hyperbola (4.15), its position uniquely defined by the parameters (t_1, t_2, t_3^2) of the translation matrix $\mathbf{T}^{\epsilon E}$. Changing the parameters (t_1, t_2, t_3^2) is equivalent to moving and resizing the set Ω_T . Note that the coefficient in front of the ‘main’ (bilinear) term $y(\sigma_*) \cdot y(\kappa_*)$ of this hyperbola is positive. The conditions (4.10) are equivalent to saying that the points $(-\sigma_i, -\kappa_i)$, $i = 1, 2$ belong to the set Ω_T (cf. (4.10) and (4.15)). From (4.11) it follows that $(\sigma_i, \mu_i) \in \Omega_T$ for $i = 1, 2$. The conditions (4.12) mean that $(-\sigma_i, 0) \in \Omega_T$ for $i = 1, 2$. These are the only restrictions on the set of the parameters (t_1, t_2, t_3^2) , i.e. on the position and the size of the set Ω_T . Hence, any pair $(y(\sigma_*), y(\kappa_*))$ of a composite should belong to the intersection of all the sets Ω_T that contain the points $(-\sigma_i, 0)$, $i = 1, 2$, the points $(-\sigma_i, -\kappa_i)$, $i = 1, 2$ and the points (σ_i, μ_i) , $i = 1, 2$. One can check that the conditions (4.12) are more restrictive than (4.10).

In summary, there exists a one-to-one correspondence between the triples of the parameters (t_1, t_2, t_3^2) and the hyperbolas with positive coefficient in front of the term $y(\sigma_*) \cdot y(\kappa_*)$ in the $y(\sigma_*)$ - $y(\kappa_*)$ plane. By changing these parameters one can move and deform the hyperbola in that plane. The parameters have to be chosen so that

all of the aforementioned points lie inside the set Ω_T and make the bound (4.15) the most restrictive one. Analysis of the bounds (4.15) and restrictions (4.10)–(4.12) for any composite with badly ordered phases leads to the following bounds (see figure 4):

Statement 4.1. The lower bound on the set of pairs $(y(\sigma_*), y(\kappa_*))$ in the plane $y(\sigma_*)-y(\kappa_*)$ is given by the lowest of the two hyperbolas

$$\text{Hyp}[(\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_1, 0)], \quad \text{Hyp}[(\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_2, 0)].$$

Remark 4.1. Condition (4.2) guarantees the existence of the parameters t_1, t_2, t_3 such that the bounding hyperbola with the positive coefficient in front of the bilinear term passes through the points $(\sigma_1, \mu_1), (\sigma_2, \mu_2)$, simultaneously. In this case the conditions (4.1), $i = 1, 2$, are equalities and define two equations for the three parameters t_1, t_2, t_3 . The strongest one among the conditions (4.12) defines the third equality that allows one to define all of the coefficients. One can analyse the conditions (4.12) in order to find the strongest one. For example, it is clear that the inequality (4.12) with $i = 1$ is stronger than (4.12) with $i = 2$ if $\sigma_1 \leq \sigma_2$. However, we avoid such analyses and use the bounds in the form described above.

Let us now examine the functional that is conjugate to the functional $I_{\epsilon E}(\epsilon_h, \mathbf{E}_h)$, namely,

$$I_{\tau J}(\boldsymbol{\tau}_h, \mathbf{J}_h) = \boldsymbol{\tau}_h : \mathbf{S}_* : \boldsymbol{\tau}_h + \mathbf{J}_h : \hat{\boldsymbol{\Sigma}}_*^{-1} : \mathbf{J}_h. \quad (4.16)$$

It can be written in the quadratic form,

$$I_{\tau J}(\boldsymbol{\tau}_h, \mathbf{J}_h) = \mathbf{e}_0 \cdot \mathbf{D}_*^{\tau J} \cdot \mathbf{e}_0, \quad (4.17)$$

where $\mathbf{e}_0 = \langle \mathbf{e}_0 \rangle$ is seven-dimensional vector composed of the components of the stress field $\boldsymbol{\tau}$ and matrix of current fields \mathbf{J} in the basis $\mathbf{a}_1-\mathbf{a}_4$ as

$$\mathbf{e} = [\boldsymbol{\tau} : \mathbf{a}_1, \boldsymbol{\tau} : \mathbf{a}_2, \boldsymbol{\tau} : \mathbf{a}_3, \mathbf{J} : \mathbf{a}_1, \mathbf{J} : \mathbf{a}_2, \mathbf{J} : \mathbf{a}_3, \mathbf{J} : \mathbf{a}_4]. \quad (4.18)$$

The matrix $\mathbf{D}_*^{\tau J}$ has the block diagonal form,

$$\mathbf{D}_*^{\tau J} = \begin{pmatrix} \mathbf{S}_* & 0 \\ 0 & \hat{\boldsymbol{\Sigma}}_*^{-1} \end{pmatrix}. \quad (4.19)$$

As follows from (3.45), (3.49), (3.51), the quadratic form

$$\psi_{\tau J} = \mathbf{e} \cdot \mathbf{T}^{\tau J} \cdot \mathbf{e} \quad (4.20)$$

is quasi-affine, if the vector \mathbf{e} is given by (4.18) and the matrix $\mathbf{T}^{\tau J}$ is chosen to be the block matrix in the form,

$$\mathbf{T}^{\tau J} = \begin{pmatrix} \boldsymbol{\Phi}_{\tau\tau}(-t_1, 0, 0, 0) & \boldsymbol{\Phi}_{\tau J}(-t_3, 0, 0, 0) \\ \boldsymbol{\Phi}_{\tau J}^T(-t_3, 0, 0, 0) & \boldsymbol{\Phi}_{JJ}(-t_2, 0, 0, 0) \end{pmatrix}. \quad (4.21)$$

For an isotropic composite with isotropic phases, the bounds (3.37) and the restrictions (3.34) are written as follows:

$$\mathbf{Y}(\mathbf{D}_*^{\tau J}) + \mathbf{T}^{\tau J} \geq 0 \quad (4.22)$$

when

$$\mathbf{D}_i^{\tau J} - \mathbf{T}^{\tau J} \geq 0. \quad (4.23)$$

Here the matrices $\mathbf{Y}(\mathbf{D}_*^{\tau J}) + \mathbf{T}^{\tau J}$ and $\mathbf{D}_i^{\tau J} - \mathbf{T}^{\tau J}$ have a block structure and can be written as a direct sum of subblocks,

$$\mathbf{Y}(\mathbf{D}_*^{\tau J}) + \mathbf{T}^{\tau J} = \mathbf{A}_*^{1,4} \oplus \mathbf{A}_*^{2,5} \oplus \mathbf{A}_*^{3,6} \oplus \mathbf{A}_*^7, \quad (4.24)$$

$$\mathbf{D}_i^{\tau J} - \mathbf{T}^{\tau J} = \mathbf{A}_i^{1,4} \oplus \mathbf{A}_i^{2,5} \oplus \mathbf{A}_i^{3,6} \oplus \mathbf{A}_i^7, \quad (4.25)$$

where we use the same notation for the matrices $\mathbf{A}_i^{k,l}$ as a diagonal minors of the corresponding matrices. The bounds come from the 'main' block,

$$\det \mathbf{A}_*^{1,4} = (y(1/2\kappa_*) - t_1)(y(1/\sigma_*) - t_2) - t_3^2 \geq 0; \quad (4.26)$$

the restrictions on the parameters t_1, t_2, t_3 have the form,

$$\det \mathbf{A}_i^{1,4} = (1/2\kappa_i + t_1)(1/\sigma_i + t_2) - t_3^2 \geq 0, \quad i = 1, 2, \quad (4.27)$$

$$\det \mathbf{A}_i^{2,5} = \det \mathbf{A}_i^{3,6} = (1/2\mu_i - t_1)(1/\sigma_i - t_2) - t_3^2 \geq 0, \quad i = 1, 2. \quad (4.28)$$

Let us define, in a manner similar as was done earlier, the set Ω_T as a set of pairs $(1/\kappa_*, 1/\sigma_*)$ that satisfy the inequality (4.26) for some fixed values of the parameters (t_1, t_2, t_3) . Conditions (4.27) mean that the points $(-1/\sigma_i, -1/\kappa_i)$ belong to the set Ω_T . Similarly, inequalities (4.28) require the points $(1/\sigma_i, 1/\mu_i)$ to lie inside the set Ω_T . The boundary of the set Ω_T is a hyperbola (defined by the equality of equation (4.26)) with positive coefficient in front of the term $1/\sigma_* \cdot 1/\kappa_*$ in the $1/\sigma_* - 1/\kappa_*$ plane. Changing the parameters t_i is equivalent to moving and resizing the set Ω_T . In order to find the tightest bounds, one needs to find the intersections of all such sets Ω_T that contain the points $(1/\sigma_i, 1/\mu_i), (-1/\sigma_i, -1/\kappa_i), i = 1, 2$. The result for a composite with badly ordered phases (see (4.2)) is given by the following statement:

Statement 4.2. The lower bound on the set of pairs $(y(1/\sigma_*), y(1/\kappa_*))$ in the $y(1/\sigma_*) - y(1/\kappa_*)$ plane is given by the lowest of the two hyperbolas

$$\text{Hyp} \left[\left(\frac{1}{\sigma_1}, \frac{1}{\mu_1} \right), \left(\frac{1}{\sigma_2}, \frac{1}{\mu_2} \right), \left(-\frac{1}{\sigma_1}, -\frac{1}{\kappa_1} \right) \right],$$

$$\text{Hyp} \left[\left(\frac{1}{\sigma_1}, \frac{1}{\mu_1} \right), \left(\frac{1}{\sigma_2}, \frac{1}{\mu_2} \right), \left(-\frac{1}{\sigma_2}, -\frac{1}{\kappa_2} \right) \right].$$

Note that due to the properties of Y -transformation, one has $y(1/\sigma_*) = 1/y(\sigma_*)$ and $y(1/2\kappa_*) = 1/2y(\kappa_*)$. Therefore, hyperbolas in the $y(1/\sigma_*) - y(1/\kappa_*)$ plane correspond to hyperbolas in the $y(\sigma_*) - y(\kappa_*)$ plane. Statement 4.2 can be reformulated as follows:

Statement 4.3. The upper bound on the set of pairs $(y(\sigma_*), y(\kappa_*))$ in the plane $y(\sigma_*) - y(\kappa_*)$ is given by the highest of the two hyperbolas

$$\text{Hyp}((\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_1, -\kappa_1)), \quad \text{Hyp}((\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_2, -\kappa_2)).$$

It is now seen that the aforementioned statements 4.1 and 4.3 can be combined as follows:

Statement 4.4. In order to find bounds on the set of pairs $(y(\sigma_*), y(\kappa_*))$ for any composite with badly ordered phases, one should inscribe in the $y(\sigma_*) - y(\kappa_*)$ plane the four following segments of the hyperbolas:

$$\text{Hyp}((\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_1, -\kappa_1)), \quad \text{Hyp}((\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_2, -\kappa_2)),$$

$$\text{Hyp}((\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_1, 0)), \quad \text{Hyp}((\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_2, 0)).$$

The outermost two of these curves represent the required bounds.

Now we need to transform the bounds into the plane of the actual moduli, not their Y -transformations. First we mention that Y -transformation is a fractional-linear one. Therefore, hyperbolas in the $y(\sigma_*)-y(\kappa_*)$ plane correspond to the hyperbolas in the $\sigma_*-\kappa_*$ plane. Any hyperbola can be defined by three points that it passes through. Hence, in order to transform the results into the plane of actual moduli, we need to study the correspondence between the characteristic points on the boundary hyperbolas. We note that

$$y(\sigma_{i*}) = \sigma_i, \quad y(\sigma_i) = -\sigma_i, \quad y(\kappa_{i*}) = \mu_i, \quad y(\kappa_i) = -\kappa_i, \quad i = 1, 2; \quad y(0) = \kappa_h, \quad (4.29)$$

where the values σ_{i*} , κ_{i*} $i = 1, 2$ and κ_h are defined by the equations (1.4), (1.5), and (1.8), respectively. Therefore, statement 4.4 is equivalent to the statement 1.1 in the specific case (4.2) of badly ordered materials.

(b) *Well-ordered materials*

The proof of the statement 1.1 for the composite of two well-ordered materials is almost identical to the badly ordered case. Therefore, we omit most of the details and only mention that the lower bound on bulk modulus follows from bounds on the functional,

$$I_{\epsilon J}(\epsilon_h, \mathbf{J}_h) = \epsilon_h : \mathbf{C}_* : \epsilon_h + \mathbf{J}_h : \hat{\Sigma}_*^{-1} : \mathbf{J}_h = \zeta_h : \mathbf{C}'_* : \zeta_h + \mathbf{J}_h : \hat{\Sigma}_*^{-1} : \mathbf{J}_h = \mathbf{e}_0 \cdot \mathbf{D}_*^{\epsilon J} \cdot \mathbf{e}_0, \quad (4.30)$$

where

$$\mathbf{e}_0 = \langle \mathbf{e} \rangle, \quad \mathbf{e} = [\zeta : \mathbf{a}_1, \zeta : \mathbf{a}_2, \zeta : \mathbf{a}_3, \zeta : \mathbf{a}_4, \mathbf{J} : \mathbf{a}_1, \mathbf{J} : \mathbf{a}_2, \mathbf{J} : \mathbf{a}_3, \mathbf{J} : \mathbf{a}_4], \quad (4.31)$$

and

$$\mathbf{D}_*^{\epsilon J} = \begin{pmatrix} \mathbf{C}'_* & 0 \\ 0 & \hat{\Sigma}_*^{-1} \end{pmatrix}. \quad (4.32)$$

We should use the the quadratic form

$$\mathbf{T}^{\zeta J} = \begin{pmatrix} \Phi_{\zeta\zeta}(-t_1, 0, 0, 0) & \Phi_{\zeta J}(-t_3, 0, 0, 0) \\ \Phi_{\zeta J}^T(-t_3, 0, 0, 0) & \Phi_{JJ}(-t_2, 0, 0, 0) \end{pmatrix}. \quad (4.33)$$

Quasi-convexity of this form follows directly from the conditions (3.40), (3.45), and (3.46). The result is given by the following statement:

Statement 4.5. The lower bound on the set of the pairs $(y(1/\sigma_*), y(\kappa_*))$ for any composite with two well-ordered phases (see (4.1)) is given, in the $y(1/\sigma_*)-y(\kappa_*)$ plane, by the lowest of the two hyperbolas

$$\text{Hyp}[(1/\sigma_1, \mu_1), (1/\sigma_2, \mu_2), (-1/\sigma_1, -\kappa_1)], \quad \text{Hyp}[(1/\sigma_1, \mu_1), (1/\sigma_2, \mu_2), (-1/\sigma_2, -\kappa_2)].$$

One can check that hyperbolas in the $y(1/\sigma_*)-y(\kappa_*)$ plane correspond to the hyperbolas in the $y(\sigma_*)-y(\kappa_*)$ plane. Therefore, we have following bound:

Statement 4.6. The lower bound on the set of the pairs $(y(\sigma_*), y(\kappa_*))$ for any composite with two well-ordered phases (see (4.1)) is given, in the $y(\sigma_*)-y(\kappa_*)$ plane, by the lowest of the two hyperbolas

$$\text{Hyp}[(\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_1, -\kappa_1)], \quad \text{Hyp}[(\sigma_1, \mu_1), (\sigma_2, \mu_2), (-\sigma_2, -\kappa_2)].$$

As can be seen, the above proof is almost identical to the proof for the badly ordered case with the obvious interchange of the symbols σ_i and $1/\sigma_i$. The upper bound can be obtained in a similar manner by studying the functional $I_{\tau E}(\boldsymbol{\tau}_h, \mathbf{E}_h)$. Combining lower and upper bounds we end up with the set of bounds in the $y(\sigma_*)-y(\kappa_*)$ plane that coincides with the statement 4.4 and equivalent to statement 1.1 for the well-ordered case. This completes the proof of statement 1.1.

5. Coupled conductivity-shear modulus bounds

Here we shall prove the new bounds that couple the effective conductivity with the effective shear modulus of any isotropic, two-phase composite at fixed volume fraction f_1 , i.e. statement 1.2. In this Section we call the materials well-ordered if $(\sigma_1 - \sigma_2)(\kappa_1 - \kappa_2) \geq 0$ and badly ordered if $(\sigma_1 - \sigma_2)(\kappa_1 - \kappa_2) \leq 0$.

(a) Badly ordered materials

(i) Lower bound

Let us first estimate the functional $I_{\zeta\zeta E}(\zeta_s^{(1)}, \zeta_s^{(2)}, \mathbf{E}_h)$ which allows one to obtain the coupled bound, as was mentioned above in §3. For this functional the vector \mathbf{e} is 12-dimensional vector with the components:

$$e_i = \zeta_s^{(1)} : \mathbf{a}_i, \quad e_{4+i} = \zeta_s^{(2)} : \mathbf{a}_i, \quad e_{8+i} = \mathbf{E}_i : \mathbf{a}_i, \quad i = 1, 2, 3, 4. \quad (5.1)$$

The matrices $\mathbf{D}_i^{\zeta\zeta E}$, $i = 1, 2$ and $\mathbf{D}_*^{\zeta\zeta E}$ have the following block-diagonal form:

$$\mathbf{D}_i = \mathbf{D}_i^{\zeta\zeta E} = \begin{pmatrix} \mathbf{C}'_* & 0 & 0 \\ 0 & \mathbf{C}'_i & 0 \\ 0 & 0 & \hat{\Sigma}_i \end{pmatrix}, \quad \mathbf{D}_* = \mathbf{D}_*^{\zeta\zeta E} = \begin{pmatrix} \mathbf{C}'_* & 0 & 0 \\ 0 & \mathbf{C}'_* & 0 \\ 0 & 0 & \hat{\Sigma}_* \end{pmatrix}. \quad (5.2)$$

We use the translation matrix $\mathbf{T}^{\zeta\zeta E}(t_1, t_2, t_3, t_4)$ with the block form,

$$\mathbf{T}^{\zeta\zeta E} = \begin{pmatrix} \boldsymbol{\Phi}_{\zeta\zeta}(-t_1, 0, 0, 0) & \boldsymbol{\Phi}_{\zeta\zeta}(0, 0, 0, -t_3) & \boldsymbol{\Phi}_{\zeta\zeta}(0, -t_4, 0, 0) \\ \boldsymbol{\Phi}_{\zeta\zeta}^T(0, 0, 0, -t_3) & \boldsymbol{\Phi}_{\zeta\zeta}(-t_1, 0, 0, 0) & \boldsymbol{\Phi}_{\zeta\zeta}(0, 0, -t_4, 0) \\ \boldsymbol{\Phi}_{\zeta\zeta}^T(0, -t_4, 0, 0) & \boldsymbol{\Phi}_{\zeta\zeta}^T(0, 0, -t_4, 0) & \boldsymbol{\Phi}_{\zeta\zeta}(-t_2, 0, 0, 0) \end{pmatrix}, \quad (5.3)$$

where $\boldsymbol{\Phi}_{\zeta\zeta}$ is defined by (3.40). The restrictions (3.34) have the form,

$$\mathbf{D}_i^{\zeta\zeta E} - \mathbf{T}^{\zeta\zeta E}(t_1, t_2, t_3, t_4) \geq 0, \quad i = 1, 2. \quad (5.4)$$

We first mention that each of the matrices $[\mathbf{D}_i^{\zeta\zeta E} - \mathbf{T}^{\zeta\zeta E}]$ ($i = 1, 2$) can be written as a sum of four blocks, namely,

$$\mathbf{D}_i^{\zeta\zeta E} - \mathbf{T}^{\zeta\zeta E} = \mathbf{A}_i^{1,8,10} \oplus \mathbf{A}_i^{2,7,9} \oplus \mathbf{A}_i^{3,6,12} \oplus \mathbf{A}_i^{4,5,11}. \quad (5.5)$$

Again the matrix $\mathbf{A}_i^{a,b,c}$ here denotes a diagonal minor of the matrix $[\mathbf{D}_i^{\zeta\zeta E} - \mathbf{T}^{\zeta\zeta E}]$; it consists from elements standing on the intersection of lines and columns with numbers a, b, c .

The matrices on the right-hand side of (5.5) have the following form:

$$\mathbf{A}_i^{1,8,10} = \begin{pmatrix} 2\kappa_i + t_1 & t_3 & t_4 \\ t_3 & t_1 & -t_4 \\ t_4 & -t_4 & \sigma_i - t_2 \end{pmatrix}, \quad \mathbf{A}_i^{2,7,9} = \begin{pmatrix} 2\mu_i - t_1 & t_3 & -t_4 \\ t_3 & 2\mu_i - t_1 & -t_4 \\ -t_4 & -t_4 & \sigma_i + t_2 \end{pmatrix}, \quad (5.6)$$

$$\mathbf{A}_i^{3,6,12} = \begin{pmatrix} 2\mu_i - t_1 & -t_3 & -t_4 \\ -t_3 & 2\mu_i - t_1 & t_4 \\ -t_4 & t_4 & \sigma_i + t_2 \end{pmatrix}, \quad \mathbf{A}_i^{4,5,11} = \begin{pmatrix} t_1 & -t_3 & t_4 \\ -t_3 & 2\kappa_i + t_1 & t_4 \\ t_4 & t_4 & \sigma_i - t_2 \end{pmatrix}. \quad (5.7)$$

Parameters t_i should satisfy the relations,

$$\det \mathbf{A}_i^{2,7,9} = \det \mathbf{A}_i^{3,6,12} \geq 0, \quad i = 1, 2; \quad (5.8)$$

$$\det \mathbf{A}_i^{1,8,10} = \det \mathbf{A}_i^{4,5,11} \geq 0, \quad i = 1, 2. \quad (5.9)$$

For the isotropic composite, the bound (3.37) becomes

$$\det[\mathbf{Y}(\mathbf{D}_*^{\zeta\zeta E}) + \mathbf{T}^{\zeta\zeta E}]^{2,7,9} = [(y(\sigma_*) - t_1)(2y(\mu_*) + t_2 - t_4) - 2t_3^2(2y(\mu_*) + t_2 + t_4)] \geq 0, \quad (5.10)$$

where

$$[\mathbf{Y}^{\zeta\zeta E} + \mathbf{T}^{\zeta\zeta E}]^{2,7,9} = \begin{pmatrix} 2y(\mu_*) + t_1 & -t_3 & t_4 \\ -t_3 & 2y(\mu_*) + t_1 & t_4 \\ t_4 & t_4 & y(\sigma_*) - t_2 \end{pmatrix}. \quad (5.11)$$

For any fixed values of the parameters t_i , the condition (5.10) defines the set Ω_T in the $y(\sigma_*)-y(\mu_*)$ plane that contains the pair $(y(\sigma_*), y(\mu_*))$ for any composite. The boundary of this set is a hyperbola in the $y(\sigma_*)-y(\mu_*)$ plane with a positive coefficient in front of the main term $y(\sigma_*) \cdot y(\mu_*)$. The conditions (5.8) are equivalent to saying that this set Ω_T contains the points $(-\sigma_i, -\mu_i)$, $i = 1, 2$.

Let us put the value of t_4 as

$$t_4 = 2\mu_{\min} - t_2, \quad \mu_{\min} = \min\{\mu_1, \mu_2\}. \quad (5.12)$$

Then one can check that the conditions (5.9) require the set Ω_T to contain the points $(\sigma_i, \kappa_i \mu_{\min} / (\kappa_i + 2\mu_{\min}))$, $i = 1, 2$. Hence, there exists a one-to-one correspondence between the parameters $t_1, t_2, t_3, t_4 = \mu_{\min} - t_2$ and the position of the set Ω_T in the $y(\sigma_*)-y(\mu_*)$ plane such that: (i) the boundary of this set is a hyperbola that is defined by the equation (5.10); (ii) this set contains the points $(-\sigma_i, -\mu_i)$ and $(\sigma_i, \kappa_i \mu_{\min} / (\kappa_i + 2\mu_{\min}))$, $i = 1, 2$.

Changing the parameters t_1, t_2, t_3 is equivalent to moving and resizing of the set Ω_T . In order to find the most restrictive bounds one should find the intersection of all such sets. For a composite with two badly ordered phases this leads to the following lower bound:

Statement 5.1. The lower bound on the set of pairs $(y(\sigma_*), y(\mu_*))$ is given, in the $y(\sigma_*)-y(\mu_*)$ plane, by the lowest of the two hyperbolas

$$\text{Hyp}[(\sigma_1, \kappa_1 \mu_{\min} / (\kappa_1 + 2\mu_{\min})), (\sigma_2, \kappa_2 \mu_{\min} / (\kappa_2 + 2\mu_{\min})), (-\sigma_1, -\mu_1)],$$

$$\text{Hyp}[(\sigma_1, \kappa_1 \mu_{\min} / (\kappa_1 + 2\mu_{\min})), (\sigma_2, \kappa_2 \mu_{\min} / (\kappa_2 + 2\mu_{\min})), (-\sigma_2, -\mu_2)],$$

in conjunction with the inequality $y(\sigma_*) \geq \sigma_{\min} = \min\{\sigma_1, \sigma_2\}$.

(ii) *The upper bound*

To get the upper bound, we deal with the functional $I_{\tau\tau J}(\boldsymbol{\tau}_s^{(1)}, \boldsymbol{\tau}_s^{(1)}, \mathbf{J}_h)$. The proof of the upper bound is the same as for the lower bound. Here the vector \mathbf{e} is equal to

$$\mathbf{e} = \{\boldsymbol{\tau}_s^{(1)} : \mathbf{a}_1, \boldsymbol{\tau}_s^{(1)} : \mathbf{a}_2, \boldsymbol{\tau}_s^{(1)} : \mathbf{a}_2, \boldsymbol{\tau}_s^{(2)} : \mathbf{a}_1, \boldsymbol{\tau}_s^{(2)} : \mathbf{a}_2, \boldsymbol{\tau}_s^{(2)} : \mathbf{a}_3, \mathbf{J} : \mathbf{a}_1, \mathbf{J} : \mathbf{a}_2, \mathbf{J} : \mathbf{a}_3, \mathbf{J} : \mathbf{a}_4\}. \quad (5.13)$$

The matrices D_i , $i = 1, 2$ have the block diagonal form,

$$D_i = D_i^{\tau\tau J} = \begin{pmatrix} S_i & 0 & 0 \\ 0 & S_i & 0 \\ 0 & 0 & \hat{\Sigma}_i \end{pmatrix}, \quad i = 1, 2, \quad (5.14)$$

the matrix $T^{\tau\tau J}(t_1, t_2, t_3, t_4)$ is chosen in the following block form:

$$T^{\tau\tau J} = \begin{pmatrix} \Phi_{\tau\tau}(-t_1, 0, 0, 0) & \Phi_{\tau\tau}(0, 0, 0, -t_3) & \Phi_{\tau J}(0, -t_4, 0, 0) \\ \Phi_{\tau\tau}^T(0, 0, 0, -t_3) & \Phi_{\tau\tau}(-t_1, 0, 0, 0) & \Phi_{\tau J}(0, 0, -t_4, 0) \\ \Phi_{\tau J}^T(0, -t_4, 0, 0) & \Phi_{\tau J}^T(0, 0, -t_4, 0) & \Phi_{JJ}(-t_2, 0, 0, 0) \end{pmatrix}. \quad (5.15)$$

As in the previous case, each of the (10×10) matrices $D_i^{\tau\tau J} - T^{\tau\tau J}(t_1, t_2, t_3, t_4)$ can be represented as a direct sum of four blocks,

$$D_i^{\tau\tau J} - T^{\tau\tau J}(t_1, t_2, t_3, t_4) = A_i^{1,8} \oplus A_i^{2,6,7} \oplus A_i^{3,5,10} \oplus A_i^{4,9}, \quad (5.16)$$

where

$$\left. \begin{aligned} A_i^{1,8} = A_i^{4,9} &= \begin{pmatrix} 1/2\kappa_i + t_1 & -t_4 \\ -t_4 & 1/\sigma_i - t_2 \end{pmatrix}, \\ A_i^{2,6,7} &= \begin{pmatrix} 1/2\mu_i - t_1 & t_3 & t_4 \\ t_3 & 1/2\mu_i - t_1 & t_4 \\ t_4 & t_4 & 1/\sigma_i + t_2 \end{pmatrix}, \end{aligned} \right\} \quad (5.17)$$

$$A_i^{3,5,10} = \begin{pmatrix} 1/2\mu_i - t_1 & -t_3 & t_4 \\ -t_3 & 1/2\mu_i - t_1 & -t_4 \\ t_4 & -t_4 & 1/\sigma_i + t_2 \end{pmatrix}. \quad (5.18)$$

The values of the four parameters t_i of the translation matrix should satisfy the equations:

$$\det A_i^{1,8} = \det A_i^{4,9} \geq 0, \quad i = 1, 2, \quad (5.19)$$

$$\det A_i^{2,6,7} = \det A_i^{3,5,10} \geq 0, \quad i = 1, 2. \quad (5.20)$$

We obtain the upper bound from the inequality

$$Y(D_*^{\tau\tau J}) + T^{\tau\tau J} \geq 0. \quad (5.21)$$

This matrix is divided into four uncoupled blocks (for isotropic composite); the most restrictive bound comes from one of them, namely,

$$\det[Y(D_*^{\tau\tau J}) + T^{\tau\tau J}]^{2,6,7} = \det \begin{pmatrix} 1/2y(\mu_*) + t_1 & -t_3 & -t_4 \\ -t_3 & 1/y(\mu_*) + t_1 & -t_4 \\ -t_4 & -t_4 & 1/y(\sigma_*) - t_2 \end{pmatrix} \geq 0. \quad (5.22)$$

For the fixed values of the parameters t_i , this bound defines the set Ω_T in the $1/y(\sigma_*) - 1/y(\mu_*)$ plane and its boundary is a hyperbola in this plane with positive coefficient in front of the term $1/\sigma_* \cdot 1/\mu_*$. Due to the restrictions (5.20), it contains the points $(-1/\sigma_i, -1/\mu_i)$. Let us take

$$t_4 = 1/2\mu_{\max} - t_2, \quad \mu_{\max} = \max\{\mu_1, \mu_2\}. \quad (5.23)$$

Then the conditions (5.19) are equivalent to the statement that the set Ω_T contains the points $(1/\sigma_i, 1/\mu_{\max} + 2/\kappa_i)$, $i = 1, 2$. In order to find the most restrictive bounds we have to find the union of all such sets Ω_T . Hence, the bound is given by the following statement:

Statement 5.2. The lower bound on the set of the pairs $(1/y(\sigma_*), 1/y(\mu_*))$ for any composite with two badly ordered phases plane is given, in the $1/y(\sigma_*)-1/y(\mu_*)$, by the lowest of the two hyperbolas

$$\begin{aligned} & \text{Hyp}[(1/\sigma_1, 1/\mu_{\max} + 2/\kappa_1), (1/\sigma_2, 1/\mu_{\max} + 2/\kappa_2), -(1/\sigma_1, -1/\kappa_1)], \\ & \text{Hyp}[(1/\sigma_1, 1/\mu_{\max} + 2/\kappa_1), (1/\sigma_2, 1/\mu_{\max} + 2/\kappa_2), (-1/\sigma_2, -1/\kappa_2)]. \end{aligned}$$

in conjunction with the inequality $y(1/\sigma_*) \geq 1/\sigma_{\max} = \min\{1/\sigma_1, 1/\sigma_2\}$.

Hyperbolas in the $1/y(\sigma_*)-1/y(\mu_*)$ plane correspond to hyperbolas in the $y(\sigma_*)-y(\mu_*)$ plane. Therefore, statement 5.2 can be reformulated in the latter plane as follows:

Statement 5.3. The upper bound on the set of pairs $(y(\sigma_*), y(\mu_*))$ for any composite with two badly ordered phases is given, in the $y(\sigma_*)-y(\mu_*)$ plane, by the highest of the two hyperbolas

$$\begin{aligned} & \text{Hyp}[(\sigma_1, \kappa_1\mu_{\max}/(\kappa_1 + 2\mu_{\max})), (\sigma_2, \kappa_2\mu_{\max}/(\kappa_2 + 2\mu_{\max})), (-\sigma_1, -\mu_1)], \\ & \text{Hyp}[(\sigma_1, \kappa_1\mu_{\max}/\kappa_1 + 2\mu_{\max}), (\sigma_2, \kappa_2\mu_{\max}/(\kappa_2 + 2\mu_{\max})), (-\sigma_2, -\mu_2)], \end{aligned}$$

in conjunction with the inequality the $y(\sigma_*) \leq \sigma_{\max}$.

(iii) *Summary of the results for the badly ordered case*

By using statements 5.1 and 5.3 proved immediately above, one comes to the following theorem that describes the bounds on the set of the pairs $(y(\sigma_*), y(\mu_*))$ of Y -transformations of the effective properties of the isotropic composite:

Statement 5.4. In order to find bounds on the set of pairs $(y(\sigma_*), y(\mu_*))$ for any composite with two badly ordered phases, one should inscribe in the $y(\sigma_*)-y(\mu_*)$ plane the segments of the following four hyperbolas

$$\begin{aligned} & \text{Hyp}[(\sigma_1, \kappa_1\mu_{\min}/(\kappa_1 + 2\mu_{\min})), (\sigma_2, \kappa_2\mu_{\min}/(\kappa_2 + 2\mu_{\min})), (-\sigma_1, -\mu_1)], \\ & \text{Hyp}[(\sigma_1, \kappa_1\mu_{\min}/(\kappa_1 + 2\mu_{\min})), (\sigma_2, \kappa_2\mu_{\min}/(\kappa_2 + 2\mu_{\min})), (-\sigma_2, -\mu_2)], \\ & \text{Hyp}[(\sigma_1, \kappa_1\mu_{\max}/(\kappa_1 + 2\mu_{\max})), (\sigma_2, \kappa_2\mu_{\max}/(\kappa_2 + 2\mu_{\max})), (-\sigma_1, -\mu_1)], \\ & \text{Hyp}[(\sigma_1, \kappa_1\mu_{\max}/(\kappa_1 + 2\mu_{\max})), (\sigma_2, \kappa_2\mu_{\max}/(\kappa_2 + 2\mu_{\max})), (-\sigma_2, -\mu_2)] \end{aligned}$$

and two straight lines

$$\begin{aligned} y(\sigma_*) &= \sigma_1, & y(\mu_*) &\in [\kappa_1\mu_{\min}/(\kappa_1 + 2\mu_{\min}), \kappa_1\mu_{\max}/(\kappa_1 + 2\mu_{\max})], \\ y(\sigma_*) &= \sigma_2, & y(\mu_*) &\in [\kappa_2\mu_{\min}/(\kappa_2 + 2\mu_{\min}), \kappa_2\mu_{\max}/(\kappa_2 + 2\mu_{\max})]. \end{aligned}$$

The outermost of these curves give the desired bounds.

In order to complete the proof of the statement 1.2 we just have to study the correspondence between the $\sigma_*-\mu_*$ and $y(\sigma_*)-y(\mu_*)$ planes, as we similarly did for the bulk modulus bounds. One can show that statement 5.4 in the $y(\sigma_*)-y(\mu_*)$ plane is equivalent to the statement 1.2 in the $\sigma_*-\mu_*$ plane.

(b) *Well-ordered materials*

The procedure does not change in this case. It is as straightforward as for the conductivity-bulk modulus bounds and thus we leave it to the interested reader to check this case.

6. Applications and discussion

In this section we apply the cross-property bounds given in §1 (statements 1.1 and 1.2) to some special limiting cases of the phase properties as well as to specific microgeometries, including regular and random arrays of circular cylinders, hierarchical geometries corresponding to effective-medium theories, and checkerboard-type models.

(a) Equal phase moduli

First consider the case of a composite possessing *equal shear moduli* $\mu_1 = \mu_2 = \mu$. This is a trivial instance because both effective elastic moduli do not depend on the microstructure (see, for example, Christensen 1979) and therefore are not connected with the effective conductivity of the composite.

Let us now consider composites having *equal bulk moduli* $\kappa_1 = \kappa_2 = \kappa$. All composites possess the same effective bulk modulus $\kappa_* = \kappa$, independent of the structure (also see Christensen 1979), and hence κ_* is not connected to the effective conductivity. As follows from the expressions (1.6), (1.7), $\mu_{1*} = \mu_{3*}$ and $\mu_{2*} = \mu_{4*}$ in this case. In the $\sigma_* - \mu_*$ plane, the trapezium degenerates into a rectangle $\sigma_* \in [\sigma_{1*}, \sigma_{2*}]$, $\mu_* \in [\mu_{1*}, \mu_{2*}]$, bounded by the Hashin–Shtrikman points. Therefore, we again find no connection between the effective shear modulus and the effective conductivity.

(b) Superrigid, superconducting phase

Let assume that *one of the phases is superrigid and superconducting*, i.e. $\kappa_2/\kappa_1 = \infty$, $\mu_2/\mu_1 = \infty$, and $\sigma_2/\sigma_1 = \infty$. The boundary hyperbolas in this extreme case degenerate into straight lines and the bounds for fixed $f_1 = 1 - f_2$ simply as:

$$\sigma_* \geq \sigma_{1*}^\infty, \quad \kappa_{1*}^\infty \leq \kappa_* \leq \kappa_{1*}^\infty + \max \left[\frac{\kappa_1 + \mu_1}{2\sigma_1}, \frac{2\kappa_2\mu_2}{(\kappa_2 + \mu_2)\sigma_2} \right] (\sigma_* - \sigma_{1*}^\infty), \quad (6.1)$$

$$\sigma_* \geq \sigma_{1*}^\infty, \quad \mu_{1*}^\infty \leq \mu_* \leq \mu_{4*}^\infty + \max \left[\frac{\kappa_1 + 2\mu_1}{4\sigma_1}, \frac{\kappa_2\mu_2}{(\kappa_2 + \mu_2)\sigma_2} \right] (\sigma_* - \sigma_{1*}^\infty), \quad (6.2)$$

where

$$\sigma_{1*}^\infty = \frac{1 + f_2}{f_1} \sigma_1, \quad \kappa_{1*}^\infty = \frac{\kappa_1 + f_2\mu_1}{f_1}, \quad \mu_{1*}^\infty = \frac{(1 + f_2)\kappa_1\mu_1 + 2\mu_1^2}{f_1(\kappa_1 + 2\mu_1)}, \quad \mu_{4*}^\infty = \frac{f_2\kappa_1 + 2\mu_1}{2f_1} \quad (6.3)$$

Note that the lower bounds on the elastic moduli are independent of the conductivity and coincide with the corresponding Hashin–Shtrikman lower bounds. As easily follows from the equations (6.1)–(6.2), for arbitrary $f_1 = 1 - f_2$, the following relations hold:

$$\sigma_* \geq \sigma_1, \quad \kappa_1 \leq \kappa_* \leq \kappa_1 + \max \left[\frac{\kappa_1 + \mu_1}{2\sigma_1}, \frac{2\kappa_2\mu_2}{(\kappa_2 + \mu_2)\sigma_2} \right] (\sigma_* - \sigma_1), \quad (6.4)$$

$$\sigma_* \geq \sigma_1, \quad \mu_1 \leq \mu_* \leq \mu_1 + \max \left[\frac{\kappa_1 + 2\mu_1}{4\sigma_1}, \frac{\kappa_2\mu_2}{(\kappa_2 + \mu_2)\sigma_2} \right] (\sigma_* - \sigma_1). \quad (6.5)$$

At first glance it appears odd that the bounds can depend on the ratio of the infinite moduli of the ideal phase. This occurs because a very small amount (volume fraction f_2 of order $1/\kappa_2$ or $1/\sigma_2$) of a very rigid, conducting material can yield finite effective properties.

The upper bounds defined by each of these equations represent straight lines whose

slopes depend on the ratios of the quantities under the maximum operation. The lower bounds are trivial and coincide with Hashin–Shtrikman bounds for κ_* . These bounds are *optimal* because there exist composites (namely, polycrystals made from laminate composites) that possess finite κ_* but infinite σ_* . The upper bounds are also optimal since they correspond to singly coated-circle assemblages (bound (6.4)) or doubly coated-circle assemblages (bound (6.1)) having an outermost concentric shell composed of the phase that determine the slope of the bound. The attainability of the upper bound on μ_* is still an open question.

(c) *Perfectly insulating void phase*

Let us now assume that *one of the phases is composed of voids*, i.e. $\kappa_2/\kappa_1 = 0$, $\mu_2/\mu_1 = 0$, $\sigma_2/\sigma_1 = 0$. It is convenient to present the results in the inverse coordinates, i.e. in the $1/\sigma_* - 1/\kappa_*$ and $1/\sigma_* - 1/\mu_*$ planes. For a fixed volume fraction, the bounds simply as

$$\frac{1}{\sigma_*} \geq \frac{1}{\sigma_{1*}^0}, \quad \frac{1}{\kappa_*} \geq \frac{1}{\kappa_{1*}^0} + \min \left[\frac{(\kappa_1 + \mu_1)\sigma_1}{2\kappa_1\mu_1}, \frac{2\sigma_2}{\kappa_2 + \mu_2} \right] \left(\frac{1}{\sigma_*} - \frac{1}{\sigma_{1*}^0} \right), \quad (6.6)$$

$$\frac{1}{\sigma_*} \geq \frac{1}{\sigma_{1*}^0}, \quad \frac{1}{\mu_*} \geq \frac{1}{\mu_{1*}^0} + \min \left[\frac{(\kappa_1 + \mu_1)\sigma_1}{\kappa_1\mu_1}, \frac{4\sigma_2}{\kappa_2 + 2\mu_2} \right] \left(\frac{1}{\sigma_*} - \frac{1}{\sigma_{1*}^0} \right), \quad (6.7)$$

where

$$\frac{1}{\sigma_{1*}^0} = \frac{1 + f_2}{f_1\sigma_1}, \quad \frac{1}{\kappa_{1*}^0} = \frac{\mu_1 + f_2\kappa_1}{f_1\kappa_1\mu_1}, \quad \frac{1}{\mu_{1*}^0} = \frac{(1 + f_2)\kappa_1 + 2f_2\mu_1}{f_1\kappa_1\mu_1}. \quad (6.8)$$

For arbitrary volume fractions, the bounds are given by

$$\frac{1}{\sigma_*} \geq \frac{1}{\sigma_1}, \quad \frac{1}{\kappa_*} \geq \frac{1}{\kappa_1} + \min \left[\frac{(\kappa_1 + \mu_1)\sigma_1}{2\kappa_1\mu_1}, \frac{2\sigma_2}{\kappa_2 + \mu_2} \right] \left(\frac{1}{\sigma_*} - \frac{1}{\sigma_1} \right), \quad (6.9)$$

$$\frac{1}{\sigma_*} \geq \frac{1}{\sigma_1}, \quad \frac{1}{\mu_*} \geq \frac{1}{\mu_1} + \min \left[\frac{(\kappa_1 + \mu_1)\sigma_1}{\kappa_1\mu_1}, \frac{4\sigma_2}{\kappa_2 + 2\mu_2} \right] \left(\frac{1}{\sigma_*} - \frac{1}{\sigma_1} \right). \quad (6.10)$$

The bounds on κ_* are also optimal for reasons similar to previous case. All lower bounds are trivial and realizable by some polycrystal constructions.

Inequalities (6.6)–(6.7) and (6.9)–(6.10) lead to the following bounds on the Young modulus E_*

$$\frac{1}{\sigma_*} \geq \frac{1}{\sigma_0}, \quad \frac{1}{E_*} \geq \frac{1}{E_0} + \frac{\sigma_1}{2E_1} \left(\frac{1}{\sigma_*} - \frac{1}{\sigma_1} \right), \quad \text{if } f_1 = 1 - f_2 \text{ is known,} \quad (6.11)$$

$$\frac{1}{\sigma_*} \geq \frac{1}{\sigma_1}, \quad \frac{1}{E_*} \geq \frac{1}{E_1} + \frac{\sigma_1}{2E_1} \left(\frac{1}{\sigma_*} - \frac{1}{\sigma_1} \right), \quad \text{for all } f_1, \quad (6.12)$$

where $E_* = 4\kappa_*\mu_*/(\kappa_* + \mu_*)$, $E_1 = 4\kappa_1\mu_1/(\kappa_1 + \mu_1)$, $E_0 = f_1E_1/(1 + 2f_2)$, and where we have assumed that phase 1 defines the slope of the bound.

Can our cross-property relations enable us to relate critical exponents for elasticity and conductivity near the percolation thresholds in the aforementioned extreme instances? Torquato (1992) used bounds (1.1) and (1.2) to show that the critical exponents for elasticity must always be greater than or equal to the critical exponent for conductivity near the connectivity threshold of a composite with a *perfectly insulating void phase*. Our new bounds cannot improve upon these results.

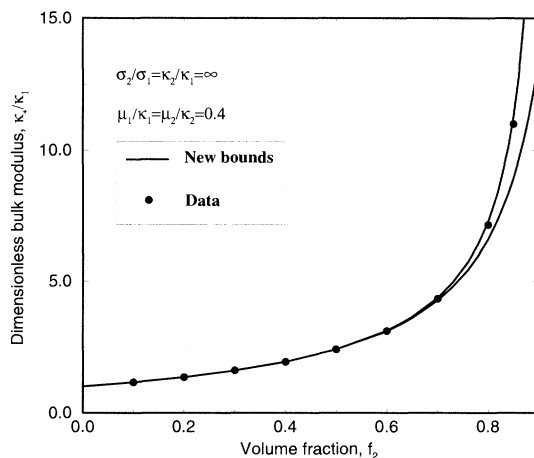


Figure 5. Comparison of the cross-property bounds on the bulk modulus (solid curves) with exact bulk modulus data by Eischen & Torquato (1993) (circles) for a superrigid, superconducting hexagonal array of circular inclusions. The bounds of statement 1.1 are calculated using exact conductivity data by Perrins *et al.* (1979).

(d) *Hexagonal and random arrays of cylinders*

How sharp are our cross-property estimates given an exact determination of one of the effective properties? To examine this question we employ exact results obtained by Perrins *et al.* (1979) for the effective conductivity and results obtained by Eischen & Torquato (1993) for the effective elastic moduli of hexagonal arrays of superconducting, superrigid inclusions (phase 2) in a matrix such that $\kappa_2/\kappa_1 = \infty$, $\mu_1/\kappa_1 = \mu_2/\kappa_2 = 0.4$, and $\sigma_2/\sigma_1 = \infty$. We make the additional but weak assumption that phase 1 determines the behaviour in relations (6.1) and (6.2), i.e.

$$\frac{\kappa_1 + \mu_1}{2\sigma_1} \geq \frac{2\kappa_2\mu_2}{(\kappa_2 + \mu_2)\sigma_2}, \quad \frac{\kappa_1 + 2\mu_1}{4\sigma_1} \geq \frac{\kappa_2\mu_2}{(\kappa_2 + \mu_2)\sigma_2}. \quad (6.13)$$

Figures 5 and 6 summarize our findings. The elastic moduli bounds (6.1) and (6.2) are calculated using the conductivity values by Perrins *et al.* (1979). Note that only the upper bounds on the elastic moduli contain conductivity information. The agreement between the bounds and the elastic-moduli data by Eischen & Torquato (1993) is quite good, especially in the case of the bulk modulus. It is important to emphasize that conventional variational upper bounds on the effective properties (such as Hashin–Shtrikman) here diverge to infinity as they do not incorporate information that the superrigid phase is in fact disconnected. In contrast, our cross-property upper bound uses the topological information that the infinite-contrast phase is disconnected through conductivity data.

Conductivity data for ‘equilibrium’ distributions of mutually impenetrable cylinders have been obtained by Kim & Torquato (1990) for several volume fractions and contrast ratios. We are not aware of elastic moduli data for the same *random* array. It is of interest to see how well our cross-property relations predict the elastic moduli in this instance. Let us consider the case of random, superconducting cylinders ($\sigma_2/\sigma_1 = \infty$) for several volume fractions and take $\kappa_2/\kappa_1 = 10$, $\mu_1/\kappa_1 = \mu_2/\kappa_2 = 0.4$.

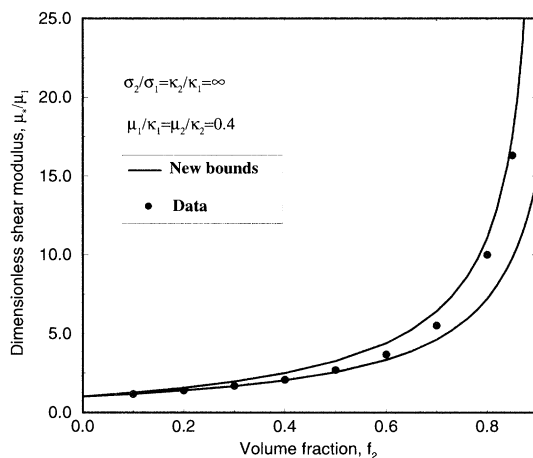


Figure 6. Comparison of the cross-property bounds on the shear modulus (solid curves) with exact shear modulus data by Eischen & Torquato (1993) (circles) for a superrigid, superconducting hexagonal array of circular inclusions. The bounds of statement 1.2 are calculated using exact conductivity data by Perrins *et al.* (1979).

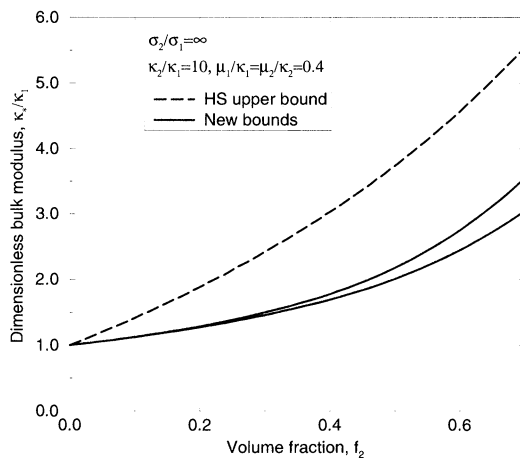


Figure 7. Cross-property bounds on the effective bulk modulus κ_* for a superconducting random array of circular inclusions with $\kappa_2/\kappa_1 = 10$, $\mu_1/\kappa_1 = \mu_2/\kappa_2 = 0.4$, given the exact effective conductivity data given by Kim & Torquato (1990). Included is the Hashin–Shtrikman upper bound. The Hashin–Shtrikman lower bound coincides with our lower bound in this case.

Note that unlike the previous example, κ_2/κ_1 is finite. Figure 7 shows the bulk modulus–conductivity bounds. One can see that they are quite sharp. Our cross-property *upper bound* provides substantial improvement over the Hashin–Shtrikman *upper bound* on κ_* which of course remains finite in this instance. At the volume fraction $f_2 = 0.7$, the Hashin–Shtrikman bound width is about 5 times larger than the cross-property bound width.

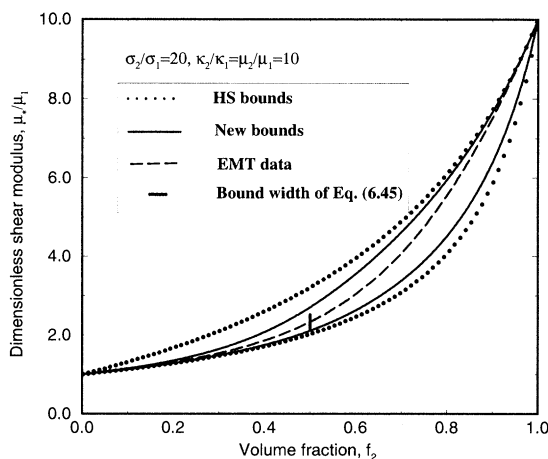


Figure 8. Comparison of the cross-property bounds on the shear modulus (solid curves) with the exact result (6.16) for the shear modulus of the effective-medium geometry. Bounds of statement 1.2 are calculated using the exact conductivity result (6.14). Dotted lines are the Hashin–Shtrikman bounds. Heavy vertical solid line at $f_1 = 0.5$ represents the optimized bound width of (6.24).

(e) *Effective-medium theory geometries*

It is useful to examine our cross-property bounds for structures in which the effective properties are known exactly analytically. One such example are the class of structures that correspond to the effective-medium theories (see Bruggeman 1935; Budiansky 1965) in which the effective properties are given by the solutions of the equations:

$$f_1 \frac{\sigma_1 - \sigma_e}{\sigma_1 + \sigma_e} + f_2 \frac{\sigma_2 - \sigma_e}{\sigma_2 + \sigma_e} = 0, \quad (6.14)$$

$$f_1 \frac{\kappa_1 - \kappa_e}{\kappa_1 + \mu_e} + f_2 \frac{\kappa_2 - \kappa_e}{\kappa_2 + \mu_e} = 0, \quad (6.15)$$

$$f_1 \frac{\mu_1 - \mu_e}{\mu_1 + \kappa_e \mu_e / (\kappa_e + 2\mu_e)} + f_2 \frac{\mu_2 - \mu_e}{\mu_2 + \kappa_e \mu_e / (\kappa_e + 2\mu_e)} = 0. \quad (6.16)$$

Milton (1984) showed that the structures that correspond to the above formulae are realized for a certain class of hierarchical granular aggregates in which grains of comparable size are well separated.

To examine our bounds for these materials, we assume that the phase properties are given by

$$\sigma_2/\sigma_1 = 20, \quad \kappa_2/\kappa_1 = \mu_2/\mu_1 = 10, \quad \kappa_1/\mu_1 = 1. \quad (6.17)$$

For a fixed volume fraction, we calculate the moduli σ_e , κ_e , and μ_e by solving the system of equations (6.14)–(6.16). Then we use the value σ_e to calculate the bounds on the effective elastic moduli of the composite, according to the statements 1.1 and 1.2 and compare the bounds with the actual values κ_e and μ_e . Figure 8 summarizes our findings for the shear modulus bounds, and includes the corresponding Hashin–Shtrikman bounds.

For $f_2 \leq 0.2$, our cross-property bounds are tight enough to provide almost exact

predictions. At larger volume fractions, our cross-property bounds provide significant improvement over the Hashin–Shtrikman bounds. The heavy solid vertical line at $f_2 = 0.5$ is the optimized bound width described immediately below.

(f) *Symmetric composite materials*

Let us now consider applying our cross-property bounds to so-called *symmetric materials*. These are two-phase composites with equal volume fractions $f_1 = f_2 = 0.5$ that are statistically topologically equivalent upon interchange of the phases. The effective conductivity σ_* of a symmetric composite is equal to (Keller 1964; Dykhne 1970)

$$\sigma_* = \sqrt{\sigma_1 \sigma_2}. \quad (6.18)$$

For such materials, however, the effective elastic moduli depend on the specific microstructure and are not known exactly for any particular cases. Given the above exact conductivity result (6.18), we can use our cross-property bounds to obtain the allowable intervals of the effective bulk and shear moduli of any such composite.

The checkerboard model (see, for example, Dykhne 1970), which has square symmetry, is one example of such a construction. The square symmetry implies that our results for the effective conductivity and bulk modulus apply for this model; our results for the shear modulus do not apply because the checkerboard is not elastically isotropic. Other examples of symmetric composites are the effective-medium geometries described above at $f_1 = f_2 = 0.5$. Still another example is a 50–50 random tessellation of space into honeycomb cells.

It is clear that the bounds on the effective elastic moduli of symmetric materials that follow from statements 1.1 and 1.2 depend on the phase conductivities σ_1 and σ_2 . One can then optimize the values σ_1 and σ_2 in order to get the tightest bounds on the effective elastic moduli. We will not go into details and only give the formulas. Specifically, if $(\kappa_1 - \kappa_2)(\mu_1 - \mu_2) \leq 0$ then

$$F_\kappa \left(\frac{2\mu_1\mu_2}{\mu_1 + \mu_2} \right) \leq \kappa_* \leq F_\kappa \left(\frac{2\mu_1\mu_2 + \kappa_{\max}(\mu_1 + \mu_2)}{\mu_1 + \mu_2 + 2\kappa_{\max}} \right), \quad (6.19)$$

$$F_\mu \left(\frac{2y_1y_3 + \mu_{\min}(y_1 + y_3)}{y_1 + y_3 + 2\mu_{\min}} \right) \leq \mu_* \leq F_\mu \left(\frac{2y_2y_4 + \mu_{\max}(y_2 + y_4)}{y_2 + y_4 + 2\mu_{\max}} \right), \quad (6.20)$$

where function F is defined by (1.3), and

$$y_1 = \frac{\kappa_1\mu_{\min}}{\kappa_1 + 2\mu_{\min}}, \quad y_2 = \frac{\kappa_2\mu_{\max}}{\kappa_2 + 2\mu_{\max}}, \quad y_3 = \frac{\kappa_2\mu_{\min}}{\kappa_2 + 2\mu_{\min}}, \quad y_4 = \frac{\kappa_1\mu_{\max}}{\kappa_1 + 2\mu_{\max}}, \quad (6.21)$$

where $\mu_{\max} = \max\{\mu_1, \mu_2\}$ and $\mu_{\min} = \min\{\mu_1, \mu_2\}$. If on the contrary

$$(\kappa_1 - \kappa_2)(\mu_1 - \mu_2) \geq 0,$$

then

$$F_\kappa \left(\frac{2\mu_1\mu_2}{\mu_1 + \mu_2} \right) \leq \kappa_* \leq F_\kappa \left(\frac{\mu_1\mu_2 + \kappa_1(\gamma\mu_1 + (1-\gamma)\mu_2)}{\gamma\mu_2 + (1-\gamma)\mu_1 + \kappa_1} \right), \quad (6.22)$$

where

$$\gamma = \frac{-(\kappa_1 + \mu_1)(\kappa_2 + \mu_2) + \sqrt{(\kappa_1 + \mu_1)(\kappa_2 + \mu_2)(\kappa_1 + \mu_2)(\kappa_2 + \mu_1)}}{(\kappa_1 - \kappa_2)(\mu_1 - \mu_2)} \quad (6.23)$$

and

$$F_{\mu} \left(\frac{y_1 y_3 + \mu_1 (\gamma' y_1 + (1 - \gamma') y_3)}{\gamma' y_3 + (1 - \gamma') y_1 + \mu_1} \right) \leq \mu_* \leq F_{\mu} \left(\frac{y_2 y_4 + \mu_1 (\gamma' y_4 + (1 - \gamma') y_2)}{\gamma' y_2 + (1 - \gamma') y_4 + \mu_1} \right), \quad (6.24)$$

where

$$\gamma' = \frac{-(y_1 + \mu_1)(y_3 + \mu_2) + \sqrt{(y_1 + \mu_1)(y_3 + \mu_2)(y_1 + \mu_2)(y_3 + \mu_1)}}{(y_1 - y_3)(\mu_1 - \mu_2)}, \quad (6.25)$$

$$\gamma'' = \frac{-(y_4 + \mu_1)(y_2 + \mu_2) + \sqrt{(y_4 + \mu_1)(y_2 + \mu_2)(y_4 + \mu_2)(y_2 + \mu_1)}}{(y_4 - y_2)(\mu_1 - \mu_2)}. \quad (6.26)$$

The optimized bounds (6.24) for the parameters given by (6.17) are represented by the bold vertical line on figure 8. These bounds provide significant improvement over our unoptimized bounds. For the bulk modulus, the optimized bounds provide a six-fold improvement over the Hashin–Shtrikman bounds. The optimized bounds are remarkably sharp, being within 8 percent of the exact results.

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